**Parabola Volume 32, Issue 2 (1996)**

# UNSW SCHOOL MATHEMATICS COMPETITION 1996

## SOLUTIONS

### JUNIOR DIVISION

1. If $x$ is a real number, $[x]$ denotes the largest integer less than or equal to $x$; for example, $[\pi] = 3$. Find all positive real numbers $x, y$ satisfying the equation

$$[x]\,[y] = x + y\ .$$

Prove that there are no solutions besides those you have found. *Solution*. Let $a = [x]$ and $b = [x]\,[y]$, and note that $a$ and $b$ are non-negative integers. From the equation we have $y = b - x$.

**Case 1.** If $x$ is an integer then so is $y$ and we have $[x] = x$, $[y] = y$. Hence $xy = x+y$ and we obtain

$$(x-1)(y-1) = xy - x - y + 1 = 1\ .$$

Therefore $x - 1 = y - 1 = 1$ and we have a solution $x = y = 2$.

**Case 2.** If $x$ is not an integer then $[y] = [b - x] = b - ([x] + 1) = b - a - 1$ and we need to solve

$$a(b - a - 1) = b\ .$$

Clearly $a$ is a factor of $b$ and we may write $b = ac$, where $c$ is an integer. Therefore

$$a(ac - a - 1) = ac\ .$$

If $a = 0$ then $b = 0$ and hence $y = -x$, which is impossible. So $a \neq 0$. Therefore $ac - a - 1 = c$ and

$$(a-1)(c-1) = ac - a - c + 1 = 2\ ,$$

which gives $a = 2, c = 3$ or $a = 3, c = 2$. Therefore $a = 2$ or $3$ and $b = 6$, and hence $2 \leq x < 4$, $y = 6 - x$. But we must remember that in this case $x$ is not an integer, so the solutions $x = 2, 3$ are to be rejected.

Putting all this back together, solutions of the given equation are (i) $2 < x < 3$ and $y = 6 - x$; (ii) $3 < x < 4$ and $y = 6 - x$; (iii) $x = y = 2$.

2. Show that if $n+1$ points are chosen at the centres of squares on an $n \times n$ chessboard, there must be two pairs of points among those chosen which are the same distance apart.

*Solution*. Let $O$ be the point in the centre of the bottom left hand square. We shall find all possible distances between pairs of points if we consider the distance from $O$ to the centres of each of the lowest two squares in the second column from the left, the lowest three in the third column, and so on up to the $n$ squares in the right hand column. So the number of different distances between the given points is at most

$$2 + 3 + \cdots + n = \frac{1}{2}(n-1)(n+2) = \frac{1}{2}(n^2 + n - 2) \ .$$

(*Comment*. The number of different distances might actually be less than this, since, for example, a distance of $5$ units horizontally is the same as a distance of $4$ units horizontally and $3$ vertically.) On the other hand, the number of ways to choose two points of the $n+1$ given points is $\frac{1}{2}n(n+1)$. Since

$$\frac{1}{2}n(n+1) = \frac{1}{2}(n^2+n) > \frac{1}{2}(n^2+n-2)$$

there are too many pairs for each of them to have a different distance; so (at least) two pairs of points must be the same distance apart.

3. Prove that for all real numbers $x, y$,

$$(x+y)^4 \geq 7x^3y + 7xy^3 + 2x^2y^2 \ .$$

*Solution*. We have

$$\begin{aligned} \text{LHS} - \text{RHS} &= (x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4) - (7x^3y + 7xy^3 + 2x^2y^2) \\ &= x^4 - 3x^3y + 4x^2y^2 - 3xy^3 + y^4 \\ &= (x-y)(x^3 - 2x^2y + 2xy^2 - y^3) \\ &= (x-y)^2(x^2 - xy + y^2) \end{aligned}$$

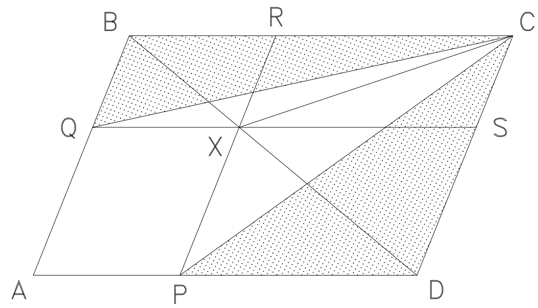
Clearly $(x-y)^2 \geq 0$; also

$$x^2 - xy + y^2 = \left(x - \frac{1}{2}y\right)^2 + \frac{3}{4}y^2 \geq 0 \ .$$

Therefore $\text{LHS} - \text{RHS} \geq 0$ and we have $(x+y)^4 \geq 7x^3y + 7xy^3 + 2x^2y^2$ for all $x, y$.

4. $ABCD$ is a parallelogram; $X$ is a point on the diagonal $BD$. A line through $X$ parallel to $AB$ intersects $AD$ at the point $P$; a line through $X$ parallel to $BC$ intersects $AB$ at $Q$. Show that the area of the quadrilateral $APCQ$ is half the area of $ABCD$.

*Solution*. Consider the following diagram. The required area is the unshaded part, which consists of a parallelogram $APXQ$ and triangles $XQC$ and $XPC$. Now triangles




$XQC$ and $XQB$ stand on the same base and lie between parallel lines, and therefore have the same area; likewise, $XPC$ and $XPD$ have the same area. Thus the unshaded area is the sum of the areas of $XQB$, $APXQ$ and $XPD$, that is, the area of $\triangle ABD$. Clearly this is half the area of $ABCD$.

*Alternative solution*. If we can show that the shaded region in the above diagram is half the area of $ABCD$, then the area of $APCQ$ will be the other half and the problem will be solved. Note, firstly, that since $PR \parallel AB$ and $QS \parallel AD$, quadrilaterals $QBRX$ and $PXSD$ are parallelograms; and, secondly, that any parallelogram is divided by a diagonal into two triangles of equal area. Therefore we have

$$\begin{aligned}\text{area}(BCD) &= \text{area}(BAD)\\ \text{area}(DPX) &= \text{area}(DSX)\\ \text{area}(BRX) &= \text{area}(BQX)\end{aligned}$$

Adding the first two of these equations and subtracting the third (refer to the diagram!) gives

$$\text{area}(CRPD) = \text{area}(AQSD)$$

and so

$$\text{area}(CPD) = \frac{1}{2}\,\text{area}(CRPD) = \frac{1}{2}\,\text{area}(AQSD)\ .$$

Thus the total shaded area is $\frac{1}{2}(\text{area}(AQSD) + \text{area}(QBCS))$, that is, half the area of $ABCD$.

5. The vertices of a regular seven-sided polygon are to be coloured red, blue, green or yellow. If a vertex is coloured red or blue then the first vertex and the fourth vertex after the red or blue vertex, counting in an anticlockwise direction, must be neither blue nor green; if a vertex is yellow or green then the first and fourth must be neither red nor yellow. Find all possible colourings of the seven vertices.

   *Solution*. Number the vertices 1,2,3,4,5,6,7 around the polygon in an anticlockwise direction. We have two pieces of information:

(a) if a vertex is red or blue then the first and fourth following vertices are neither blue nor green;

(b) if a vertex is yellow or green then the first and fourth following vertices are neither red nor yellow.

Suppose that vertex 1 is red. Then (1) tells us that vertex 2 is red or yellow and that vertex 5 is red or yellow. If vertex 5 is yellow then (2) says that the fourth vertex around from 5 may not be red or yellow; but the fourth vertex from 5 is vertex 2, which we know *is* red or yellow, so this is impossible. This argument shows that if vertex 1 is red then so is vertex 5.

Applying the same argument to vertex 5 shows that if vertex 1 is red then so are vertices 5,2,6,3,7 and 4. Clearly the same reasoning works if we start from any vertex other than 1. Therefore we have shown that if any vertex is red, then all vertices are red. A similar argument will show that if any vertex is green, then all vertices are green.

Is there any possible colouring of the vertices without using red or green? If so, the requirements of the problem become

(c) if a vertex is blue then the first and fourth following vertices are yellow;

(d) if a vertex is yellow then the first and fourth following vertices are blue.

But by an argument like the above (try it!) we may see that if vertex 1 is blue then vertex 2 must be both blue and yellow, which is impossible. Thus vertex 1 in fact is not blue, and similarly is not yellow. Therefore there are only two possible colourings for the seven vertices: all red, or all green.

6. Find all positive integers $n$ for which all of the numbers

$$n\,,\ 2n-1\,,\ 2n+5\,,\ 3n-2\,,\ 5n-4\,,\ 6n-5\,,\ \text{and } 12n+5$$

are prime. (**Note**: $1$ is not a prime.)

*Solution*. Consider the possible remainders when $n$ is divided by $7$.

- If there is no remainder then $n$ is divisible by $7$, and is therefore prime only if $n = 7$.
- If the remainder is $1$ we can write $n = 7k + 1$; then $2n + 5 = 14k + 7$, which is divisible by $7$. Therefore $2n + 5$ is prime only if $14k + 7 = 7$, that is, $k = 0$ and $n = 1$.
- If the remainder is $2$ we have $n = 7k + 2$ and so $6n - 5 = 42k + 7$, which is prime only for $k = 0$, that is, $n = 2$.
- If the remainder is $3$ then $n = 7k + 3$ and $3n - 2 = 21k + 7$, which is prime only if $k = 0$, $n = 3$.
- If the remainder is $4$ then $n = 7k + 4$, so $2n - 1 = 14k + 7$, which once again is prime only when $k = 0$ and so $n = 4$.

- If the remainder is $5$ then $n = 7k+5$ and we have $5n-4 = 35k+21 = 7(5k+3)$. This number is divisible by 7, and the quotient $5k + 3$ cannot possibly be $1$, so the number is not prime.
- If the remainder is $6$ we have $n = 7k + 6$ and so $12n + 5 = 84k + 77 = 7(12k + 11)$, which, for the same reasons as in the previous case, is never prime.

We see that in every one of the seven cases, at least one of the given expressions fails to be prime, with possible exceptions when $n = 1, 2, 3, 4$ and 7. However $n = 1$ and $n = 4$ must be ruled out as they are not prime, while $n = 2$ must also be ruled out since then $2n + 5 = 9$ which is not prime. If $n = 3$ the seven numbers are

$$3\,,\ 5\,,\ 11\,,\ 7\,,\ 11\,,\ 13\,,\ 41$$

which are indeed all prime; while if $n = 7$ then

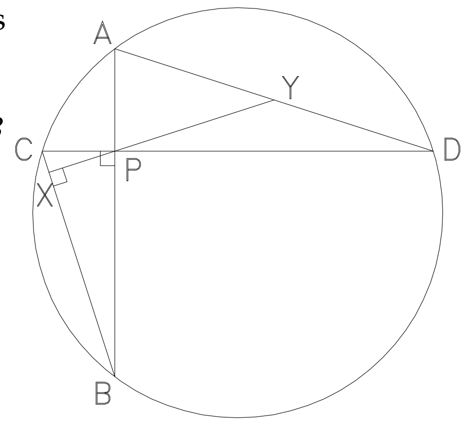
$$7\,,\ 13\,,\ 19\,,\ 19\,,\ 31\,,\ 37\,,\ 89$$

are likewise all prime. Therefore there are two possible values of $n$, namely $3$ and 7.

## SENIOR DIVISION

1. In a circle, $AB$ and $CD$ are two chords, perpendicular to each other and intersecting at $P$. The perpendicular from $P$ to $BC$ meets $BC$ at $X$. When $XP$ is extended it meets $AD$ at $Y$. Show that $Y$ is the midpoint of $AD$.

   *Solution*. In the diagram at right, triangles $BXP$ and $BPC$ are right-angled; therefore

   $$\angle XBP + \angle XPB = 90^\circ = \angle PBC + \angle PCB$$

   and so $\angle XPB = \angle PCB$. Moreover,

   $$\angle XPB = \angle YPA$$

   (vertically opposite angles), while

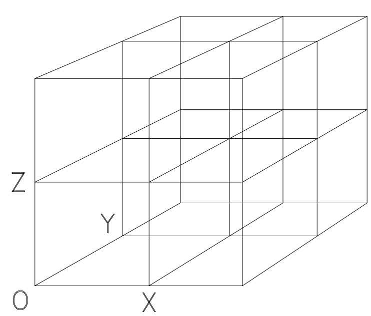
   $$\angle PCB = \angle DCB = \angle DAB = \angle YAP$$

   since $\angle DCB$ and $\angle DAB$ stand on the same arc. Hence $\angle YAP = \angle YPA$, which shows that $\triangle YAP$ is isosceles and $AY = PY$. Similarly $DY = PY$; therefore $AY = DY$ and $Y$ is the midpoint of $AD$.

2. Show that if $x, y$ and $z$ are real numbers between $0$ and $1$ then

   $$xy(1-z) + yz(1-x) + zx(1-y) \leq 1 \,.$$

   *Solution*. Consider a cube with side length $1$ and choose points $X, Y, Z$ as shown, such that $OX = x$, $OY = y$ and $OZ = z$. The cube is divided into eight smaller regions

   by three planes, parallel to the faces of the cube and passing through the points $X, Y$ and $Z$. The front top left subregion in the diagram has volume $xy(1-z)$, the front bottom right region has volume $yz(1-x)$ and the back bottom left, $zx(1-y)$. Since these three regions do not overlap, their total volume is less than the volume of the cube; that is,

   $$xy(1-z) + yz(1-x) + zx(1-y) \leq 1 \,.$$

*Alternative solution*. Since $x, y, z$ are between $0$ and $1$, the numbers

$$x\,,\ y\,,\ z\,,\ 1-x\,,\ 1-y\,,\ 1-z$$

are all non-negative. Therefore

$$\begin{aligned}
&xy(1-z)+yz(1-x)+zx(1-y)\\
&\qquad\le [xyz+xy(1-z)]+[yz(1-x)+y(1-z)(1-x)]\\
&\qquad\qquad +[zx(1-y)+(1-z)x(1-y)]\\
&\qquad\qquad +[(1-x)(1-y)z+(1-x)(1-y)(1-z)]\\
&\qquad = xy+y(1-x)+x(1-y)+(1-x)(1-y)\\
&\qquad = xy+x(1-y)+(1-x)y+(1-x)(1-y)\\
&\qquad = x+(1-x)\\
&\qquad = 1
\end{aligned}$$

*Exercise*. Explain why this is really just the same as the previous solution.

3. A row of marbles contains $n$ red and $n$ blue marbles. All the marbles are identical except for colour. How many arrangements are there in which the red marbles occur in exactly $k$ separate blocks? (For example the row

$$BRRRRBBRBRRBBB$$

contains three blocks of red marbles.)

*Solution*. First, it is obvious that if $n < k$ then the red marbles cannot be put into $k$ separate blocks, and so the number of arrangements is zero. From now on we assume that $n \ge k$. To ensure that there are $k$ separate blocks of red marbles we arrange $k$ red marbles and $k-1$ blue marbles in an alternating pattern:

$$RBRBR\ldots RBRBR\,.$$

There remain $n-k$ red and $n-k+1$ blue marbles, and we are left with the problem of how to arrange these. The $n-k$ reds have to be arranged into $k$ groups; it is permissible for some groups to be empty. Imagine the $n-k$ marbles arranged in a line, with $k-1$ dots separating them into $k$ groups. Any such arrangement is determined by choosing which $k-1$ of the $n-1$ "locations" ($n-k$ marbles and $k-1$ dots) are to hold the dots. The number of ways of choosing $k-1$ locations out of $n-1$ is given by the binomial coefficient $\binom{n-1}{k-1}$. We must also arrange the $n-k+1$ blue marbles into $k+1$ groups (the $k-1$ we have above, plus a possible extra group at each end), and a similar argument shows that this can be done in $\binom{n+1}{k}$ ways. Thus the total possible number of arrangements is

$$\binom{n-1}{k-1}\binom{n+1}{k}\,.$$

4. The vertices of a regular seven-sided polygon are to be coloured red, blue, green or yellow. If a vertex is coloured red or blue then the first vertex and the fourth vertex after the red or blue vertex, counting in an anticlockwise direction, must be neither blue nor green; if a vertex is yellow or green then the first and fourth must be neither red nor yellow. Find all possible colourings of the seven vertices.

   *Solution*. See question 5 in the Junior division.

5. An unending list of positive integers is constructed as follows. The first member of the list is chosen at random; each other number in the list is the sum of the $1996$th powers of the digits of the preceding number. Prove that there is a number which occurs at least twice in the list.

   *Solution*. Suppose that the list contains a number $n$ with $d$ digits, and let $n'$ be the next number in the list. Since the maximum possible value of $n'$ occurs when all the digits of $n$ are nines, we have

   $$n' \leq 9^{1996}d \quad \text{and} \quad n \geq 10^{d-1} .$$

   We would like to show that $n' < n$, that is,

   $$9^{1996}d < 10^{d-1} \; ; \qquad (*)$$

   in fact, this is true for $d \geq 2001$, as we shall prove by mathematical induction. If $d = 2001$ then

   $$9^{1996}d < 10^{1996} \times 10000 = 10^{2000} = 10^{d-1}$$

   and so $(*)$ is true. If $(*)$ is true for some particular $d \geq 2001$ then

   $$9^{1996}(d+1) < 9^{1996}d + 9 \times 10^{2000} < 10^{d-1} + 9 \times 10^{d-1} = 10 \times 10^{d-1} = 10^{(d+1)-1}$$

   and we see that $(*)$ is also true for $d+1$. Hence $(*)$ is true for all $d \geq 2001$.

   This inequality shows that whenever our list contains a number having more than $2000$ digits, we can be sure that the next number in the list is smaller. If this number still has more than $2000$ digits, the next will be smaller again, and so on, until we reach a number of just $2000$, or fewer, digits. Therefore, however far we calculate the list, we shall continue to find more and more numbers with $2000$ digits or fewer. But there is only a finite quantity of such numbers, and so, sooner or later, we must reach a number which has already occurred.

6. Prove that every power of $2$ has a multiple whose decimal expansion has only the digits $1$ and $2$. (For example $3 \times 2^2 = 12$ and $14 \times 2^3 = 112$.)

   *Solution*. We shall use mathematical induction to prove a bit more than is required, namely: if $k \geq 1$ then there exists $m$ such that $2^k m$ is a number consisting of $k$ digits, each of which is either $1$ or $2$.

   The result is clearly true for $k = 1$ as we may take $m = 1$.

Suppose that the result is true for some particular value of $k$, so that $n = 2^k m$ is a $k$-digit number consisting only of ones and twos. If $m$ is odd then $(5^k + m)/2$ is an integer and we have

$$2^{k+1}\,\frac{5^k + m}{2} = 10^k + 2^k m = 10\cdots 0 + n\ .$$

Since $10^k$ ends in $k$ zeros and $n$ has $k$ digits, the right hand side is a number consisting of $k + 1$ ones and twos, as required. If, on the other hand, $m$ is even, then

$$2^{k+1}\left(5^k + \frac{m}{2}\right) = 2 \times 10^k + 2^k m = 20\cdots 0 + n\ ,$$

and again this is a $k + 1$-digit number containing only the digits 1 and 2. Thus, if the statement above is true for some value of $k$ then, regardless of whether $m$ is odd or even, the statement is also true for $k + 1$.

Hence, by induction, the result is proved.

*Alternative solution*. Consider the collection $C$ of all $n$-digit numbers consisting only of ones and twos: this collection contains exactly $2^n$ numbers. Let two of the numbers in $C$ be

$$\begin{aligned} x &= 10^{n-1}a_{n-1} + 10^{n-2}a_{n-2} + \cdots + 10a_1 + a_0 \\ y &= 10^{n-1}b_{n-1} + 10^{n-2}b_{n-2} + \cdots + 10b_1 + b_0, \end{aligned}$$

where each of the digits $a_0, a_1, \ldots, a_{n-1}$ and $b_0, b_1, \ldots, b_{n-1}$ is either 1 or 2. We have

$$\begin{aligned} x - y &= 10^{n-1}(a_{n-1} - b_{n-1}) + +10^{n-2}(a_{n-2} - b_{n-2}) + \cdots + 10(a_1 - b_1) + (a_0 - b_0) \\ &= 10^{n-1}c_{n-1} + 10^{n-2}c_{n-2} + \cdots + 10c_1 + c_0, \end{aligned}$$

where every coefficient $c_k$ is 0, 1 or $-1$. Now suppose that $2^n$ is a factor of $x - y$. Since $2^n$ is not a factor of any of the numbers

$$\pm 10^{n-1},\ \pm 10^{n-2},\ \ldots\ ,\ \pm 10,\ \pm 1$$

we must have

$$c_{n-1} = c_{n-2} = \cdots = c_1 = c_0 = 0$$

and therefore $x = y$. This shows that all the $2^n$ numbers in $C$ have different remainders when divided by $2^n$; but there are only $2^n$ possible remainders, and so there must be a number for which the remainder is zero. Such a number is a multiple of $2^n$.

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### Update on Surfing the Internet

The address we gave in the last issue of Parabola for Fractals no longer exists. One that does still exist, and has Fractals (among other things) is

`http://miranda.bu.edu.cps-home.html`[1]

If you have found any other interesting addresses, please let us know.

[1] Editoral note, February 2014: this is a dead link.