Parabola Volume 32, Issue 2 (1996)

PROBLEM SECTION

You are invited to submit solutions to any or all of the following problems, accompanied by your name, school and year or form. Solutions of these problems will appear in the next issue of **Parabola**; your solution(s) may be used if they are received in time.

Hint: Some of the following problems are similar to, or based on, problems from this year's School Mathematics Competition. Solutions to the competition problems will be found elsewhere in this issue of **Parabola**.

Q.975 For which real numbers x is it true that

$$[5x] = [3x] + 2[x] + 1\ ?$$

Here $[x]$ denotes the greatest integer less than or equal to x; for example, $[\pi] = 3$.

Q.976 It was shown in problem 6 of the Senior Division that every power of 2 has a multiple whose decimal expansion contains only the digits 1 and 2. Find all pairs of non-zero digits which can replace 1 and 2 so that the statement is still true.

Q.977 Evaluate

$$\frac{1\times 2^2}{2\times 3} + \frac{2\times 2^3}{3\times 4} + \frac{3\times 2^4}{4\times 5} + \frac{4\times 2^5}{5\times 6} + \cdots + \frac{n\,2^{n+1}}{(n+1)(n+2)}.$$

Q.978 Show that three adjacent numbers in a row of Pascal's triangle can neither be in geometric progression nor in harmonic progression. (Three or more positive numbers are said to be in harmonic progression if their reciprocals are in arithmetic progression.)

Q.979 Let n be a positive integer. Find the remainder when $2^{3n} - 7n$ is divided by 49.

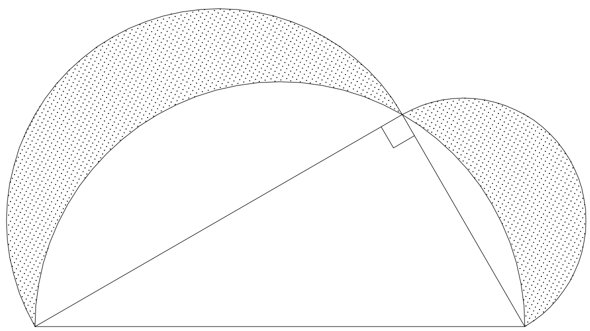
Q.980 Charlie Chump, whose algebra is not very good, believes that (by cancelling the sixes)

$$\frac{16}{64} = \frac{1\not{6}}{\not{6}4} = \frac{1}{4}$$

Find all fractions involving two-digit integers which Charlie would correctly simplify (that is, which would be reduced to lowest terms by incorrectly cancelling a digit).

Q.981 Show that a square can be cut into n smaller squares (possibly of various sizes) for any $n > 5$.

Q.982 Semicircles are drawn internally on the hypotenuse of a right-angled triangle and externally on the other two sides.



Find, without using algebra or calculus, the total area of the shaded crescents.

Q.983 Show that there are integers a, b, c, not all zero, between -10^6 and 10^6, such that

$$-10^{-11} < a + b\sqrt{2} + c\sqrt{3} < 10^{-11}.$$

Q.984 (Based on the article on page 12 of Vol. 32 No. 1). Let A, B be two sets of real numbers, each having cardinal number $\aleph_0$ and with no elements in common.

(a) Show that the cardinal number of the set

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

is $\aleph_0$.

(b) Show that the cardinal number of the set

$$\mathbb{R} - A = \{x : x \text{ is a real number and } x \notin A\}$$

is c.

(c) Show that the cardinal number of the set of all transcendental numbers is c.