Parabola Volume 32, Issue 2 (1996)

SOLUTIONS TO PROBLEMS 966-974

Q.966 Prove that

$$\binom{n}{1} - \frac{1}{2}\binom{n}{2} + \frac{1}{3}\binom{n}{3} - \cdots \pm \frac{1}{n}\binom{n}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\,,$$

where the final sign on the left hand side is $+$ if n is odd and $-$ if n is even.

ANS. The equation will be proved for all n by mathematical induction. For $n = 1$ the equation says

$$\binom{1}{1} = 1$$

which is certainly true. Now suppose that the equation is known to be true for some specific $n \geq 1$. We wish to prove that

$$\binom{n+1}{1} - \frac{1}{2}\binom{n+1}{2} + \frac{1}{3}\binom{n+1}{3} - \cdots \pm \frac{1}{n}\binom{n+1}{n} \mp \frac{1}{n+1}\binom{n+1}{n+1}$$
$$= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \frac{1}{n+1}.$$

First note that if $1 \leq k \leq n$ we have

$$\begin{aligned}
\frac{1}{k}\binom{n+1}{k} &= \frac{1}{k}\ \frac{(n+1)!}{k!\,(n+1-k)!} \\
&= \frac{n+1}{k(n+1-k)}\ \frac{n!}{k!\,(n-k)!} \\
&= \left(\frac{1}{k} + \frac{1}{n+1-k}\right)\ \frac{n!}{k!\,(n-k)!} \\
&= \frac{1}{k}\binom{n}{k} + \frac{1}{n+1}\binom{n+1}{k}.
\end{aligned}$$

Applying this formula to the previous equation gives

$$\begin{aligned}
LHS &= \binom{n}{1} + \frac{1}{n+1}\binom{n+1}{1} - \frac{1}{2}\binom{n}{2} - \frac{1}{n+1}\binom{n+1}{2} \\
&\qquad + \cdots \pm \frac{1}{n}\binom{n}{n} \pm \frac{1}{n+1}\binom{n+1}{n} \mp \frac{1}{n+1}\binom{n+1}{n+1} \\
&= \left[\binom{n}{1} - \frac{1}{2}\binom{n}{2} + \cdots \pm \binom{n}{n}\right] \\
&\qquad + \frac{1}{n+1}\left[\binom{n+1}{1} - \binom{n+1}{2} + \cdots \pm \binom{n+1}{n} \mp \binom{n+1}{n+1}\right].
\end{aligned}$$

Here the first term is

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$$

by assumption. By the Binomial Theorem,

$$(x-y)^{n+1}=x^{n+1}-\binom{n+1}{1}x^n y+\binom{n+1}{2}x^{n-1}y^2-\cdots\mp\binom{n+1}{n}xy^n\pm\binom{n+1}{n+1}y^{n+1},$$

and if we let $x=y=1$ we obtain

$$0=1-\binom{n+1}{1}+\binom{n+1}{2}-\cdots\mp\binom{n+1}{n}\pm\binom{n+1}{n+1}.$$

Taking all but the first term to the left hand side gives

$$\binom{n+1}{1}-\binom{n+1}{2}+\cdots\pm\binom{n+1}{n}\mp\binom{n+1}{n+1}=1;$$

so from above we have

$$LHS=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}+\frac{1}{n+1}=RHS,$$

which completes the proof.

ALTERNATIVE SOLUTION. (for readers who know some calculus). Edmond Park (first year, Sydney University) writes:

At the outset, consider $(1-x)^n$. This is chosen based on our observation of the binomial coefficients and alternating signs:

$$(1-x)^n=\binom{n}{0}-\binom{n}{1}x+\binom{n}{2}x^2-\cdots\pm\binom{n}{n}x^n,$$

where the final sign is $+$ if n is even, $-$ if n is odd. Now consider $\frac{1}{x}(1-x)^n$:

$$\frac{1}{x}(1-x)^n=\frac{1}{x}-\binom{n}{1}+\binom{n}{2}x-\cdots\pm\binom{n}{n}x^{n-1}$$

with the same signs as above. Now integrate both sides with respect to x, with limits 0 and 1:

$$\int_0^1\binom{n}{1}-\binom{n}{2}x+\cdots\mp\binom{n}{n}x^{n-1}dx=\int_0^1\frac{1}{x}[1-(1-x)^n]\,dx$$

where the sign is $-$ if n is even, $+$ if n is odd. Thus

$$\left[\binom{n}{1}x-\binom{n}{2}\frac{x^2}{2}+\cdots\mp\binom{n}{n}\frac{x^n}{n}\right]_0^1=\int_0^1\frac{1}{x}[x(1+(1-x)+(1-x)^2+\cdots+(1-x)^{n-1})]\,dx$$

and so

$$\begin{aligned}\binom{n}{1} - \binom{n}{2}\frac{1}{2} &+ \binom{n}{3}\frac{1}{3} - \cdots \mp \binom{n}{n}\frac{1}{n} \\ &= \int_0^1 1 + (1-x) + (1-x)^2 + \cdots + (1-x)^{n-1} dx \\ &= \left[x + \frac{(1-x)^2}{-2} + \frac{(1-x)^3}{-3} + \cdots + \frac{(1-x)^{n-1}}{-n}\right]_0^1.\end{aligned}$$

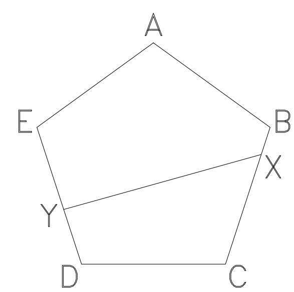
Therefore

$$\binom{n}{1} - \frac{1}{2}\binom{n}{2} + \frac{1}{3}\binom{n}{3} - \cdots \pm \frac{1}{n}\binom{n}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n},$$

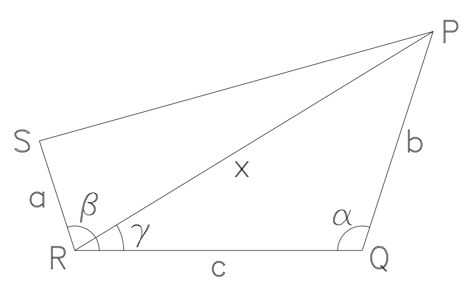
with the sign $-$ if n is even and $+$ if n is odd.

Q.967 $ABCDE$ is a regular pentagon of side length 1, and X and Y are points on BC and ED such that $BX = p$ and $EY = q$. Find q in terms of p, given that the line XY divides the pentagon into two regions of equal area.

ANS. Consider the following diagram. First we need to find the area of quadrilateral $XCDY$,



where the sides XC, CD and DY and the angles C and D are known.
First method. In the diagram following, a, b, c, α and β are known.



The area of the quadrilateral is

$$\Delta = \tfrac{1}{2}bc\sin\alpha + \frac{1}{2}ax\sin(\beta-\gamma) = \frac{1}{2}[bc\sin\alpha + ax\sin\beta\cos\gamma - ax\cos\beta\sin\gamma],$$

and we wish to eliminate the unknown quantities x and γ. By the sine rule in $\triangle PQR$ we have

$$x \sin y = b \sin \alpha \tag{1}$$

while by two applications of the cosine rule,

$$\begin{aligned} x^2 + c^2 - b^2 &= 2cx \cos \gamma \\ b^2 + c^2 - x^2 &= 2bc \cos \alpha. \end{aligned}$$

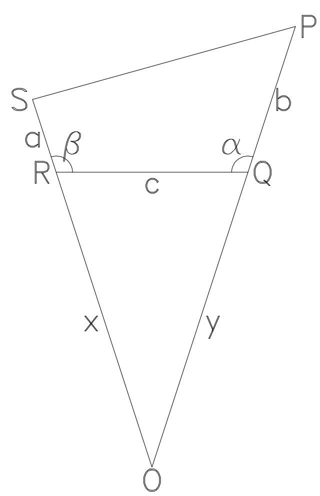
Adding these last two equations and cancelling a factor of $2c$ gives

$$c = x \cos \gamma + b \cos \alpha. \tag{2}$$

(**Exercise.** Give an alternative proof of (2) by dropping a perpendicular from P to RQ extended.) Substituting (1) and (2) into the area formula,

$$\begin{aligned} \Delta &= \frac{1}{2}\left[bc \sin \alpha + (a \sin \beta)(c - b \cos \alpha) - ab \cos \beta \sin \alpha\right] \\ &= \frac{1}{2}\left[bc \sin \alpha + ac \sin \beta - ab \sin(\alpha + \beta)\right]. \end{aligned}$$

Second method. Extend PQ and SR to meet at O; let $OR = x$ and $OQ = y$.



We have $\angle OQR = 180° - \alpha$ and $\angle ORQ = 180° - \beta$, so

$$\angle O = 180° - (180° - \alpha) - (180° - \beta) = \alpha + \beta - 180°.$$

Therefore the area of $PQRS$ is

$$\begin{aligned} \Delta &= \text{area}\,(OPS) - \text{area}\,(OQR) \\ &= \frac{1}{2}(a + x)(b + y)\sin(\alpha + \beta - 180°) - \frac{1}{2}xy \sin(\alpha + \beta - 180°). \end{aligned}$$

Using the relations

$$\frac{\sin(\alpha+\beta-180^\circ)}{c}=\frac{\sin(180^\circ-\alpha)}{x}=\frac{\sin(180^\circ-\beta)}{y}$$

found from the sine rule in $\triangle OQR$, and remembering that

$$\sin(\alpha+\beta-180^\circ)=-\sin(\alpha+\beta),\ \sin(180^\circ-\alpha)=\sin\alpha,\ \sin(180^\circ-\beta)=\sin\beta,$$

we may transform the area formula into

$$\Delta=\frac{1}{2}\left[bc\sin\alpha+ac\sin\beta-ab\sin(\alpha+\beta)\right]$$

as was found earlier.

To apply this to the pentagon problem we take $a=1-q,\ b=1-p,\ c=1$ and $\alpha=\beta=108^\circ$. This gives

$$\begin{aligned}\text{area}\,(XCDY) &= \frac{1}{2}\left[(2-p-q)\sin 108^\circ-(1-p)(1-q)\sin 216^\circ\right]\\ &= \frac{1}{2}\left[(2-p-q)\sin 72^\circ+(1-p)(1-q)\sin 36^\circ\right].\end{aligned}$$

If $p=0$ and $q=\frac{1}{2}$ we have

$$\text{area}\,(XCDY)=\frac{1}{2}\left[\frac{3}{2}\sin 72^\circ+\frac{1}{2}\sin 36^\circ\right]$$

and from the diagram it can be seen that this is half the area of the pentagon. Thus in general the relation between p and q is

$$(2-p-q)\sin 72^\circ+(1-p)(1-q)\sin 36^\circ=\frac{3}{2}\sin 72^\circ+\frac{1}{2}\sin 36^\circ.$$

Solving for q gives

$$q=\frac{(\sin 72^\circ+\sin 36^\circ)(\frac{1}{2}-p)}{\sin 72^\circ+(1-p)\sin 36^\circ}. \tag{3}$$

This can be simplified. Using the double angle formulae for sine and cosine,

$$\sin 36^\circ=2\sin 18^\circ\cos 18^\circ$$

and

$$\sin 72^\circ=2\cos 36^\circ\sin 36^\circ=4(1-2\sin^2 18^\circ)\sin 18^\circ\cos 18^\circ,$$

so that a factor of $\cos 18^\circ$ can be cancelled from numerator and denominator in (3). Also using the fact (recently proved in *Parabola*) that

$$\sin 18^\circ=\frac{-1+\sqrt{5}}{4},$$

we find after some algebraic work that

$$q = \frac{(1+\sqrt{5}) - 2(1+\sqrt{5})p}{2(1+\sqrt{5}) - 2(1-\sqrt{5})p}.$$

Finally, multiplying top and bottom by $\sqrt{5}-1$ and dividing by 4, we obtain the simpler result

$$q = \frac{1-2p}{2+(3-\sqrt{5})p}.$$

Q.968 The set of numbers $\{\,21, 24, 25, 29\,\}$ is given. It is permitted to multiply any two numbers from the set (multiplying one number by itself is also allowed) and place in the set the last two digits of the product. For example, since $21 \times 29 = 609$, we may put the number 09 into the set. By repeating this operation as often as desired, is it possible to obtain a set including the number 99?

ANS. Notice that the four given numbers are the last two digits of squares, for example

$$11^2 = 121,\ 18^2 = 324,\ 25^2 = 625,\ 27^2 = 729.$$

Take any two numbers with this property, say x and y, where $100u + x$ and $100v + y$ are squares. Then $(100u + x)(100v + y)$ is also a square. But

$$(100u + x)(100v + y) = 100(100uv + uy + vx) + xy$$

and so the last two digits of xy also form the last two digits of a square. Thus the only numbers which may ever occur in the set are those which are the last two digits of a square. However 99 is not such a number. For if $(10m + n)^2$ ends in a 9 then n must be 3 or 7; but

$$(10m + 3)^2 = 100m^2 + 60m + 9, \quad (10m + 7)^2 = 100m^2 + 140m + 49$$

and in each case the second last digit is even. Thus the set can never contain 99.

Q.969 Five positive integers add up to 1996. Find all possible values of the greatest common divisor of the five numbers.

ANS. The greatest common divisor of the five numbers will also be a factor of 1996 and therefore must be $1, 2, 4, 499, 998$ or 1996. But if the greatest common divisor is 499 or more then each of the five numbers is divisible by 499 and so their sum is at least $5 \times 499 > 1996$. Hence the only possibilities for the greatest common divisor are $1, 2$ and 4.

Q.970 Given a prime number p, find all integer solutions of the equation

$$\sqrt{x} - \sqrt{p} = \sqrt{y}\ .$$

ANS. Square both sides,

$$x - 2\sqrt{xp} + p = y,$$

rearrange,

$$2\sqrt{xp} = x + p - y,$$

and square again,

$$4xp = (x + p - y)^2.$$

Since the right hand side is a square and 4 is also a square, xp must be a square. Since p is prime we must have

$$x = pz^2$$

for some integer $z > 0$, and hence

$$4p^2z^2 = (pz^2 + p - y)^2.$$

Taking the square root of both sides and solving for y leads to

$$y = pz^2 + p \pm 2pz = p(z \pm 1)^2;$$

from the original equation it is clear that $y < x$, so the $+$ sign must be rejected and we have

$$x = pz^2,\ y = p(z-1)^2$$

where z is a positive integer.

Q.971

(a) Let m be a positive integer and n a positive integer with fewer than $m/9$ digits. Prove that if the sum of the digits of n and the sum of the digits of $2n$ are both multiples of m, then every digit of n is less than 5.

(b) Show also that if m is not a multiple of 3 then the above is true for all integers n having fewer than m digits.

ANS. Let

$$n = d_0 + 10d_1 + 10^2d_2 + \cdots + 10^kd_k,$$

where $d_0, d_1, d_2, \ldots, d_k$ are single digits. For each j write

$$2d_j = 10e_j + f_j$$

where e_j is the "carry" and f_j the "remainder". Then

$$e_j = \begin{cases} 1 & \text{if } d_j \geq 5 \\ 0 & \text{if } d_j < 5. \end{cases}$$

The sum of the digits of n is

$$d_0 + d_1 + \cdots + d_k,$$

while the sum of the digits of $2n$ is

$$\begin{aligned}(e_0 + f_0) + (e_1 + f_1) + \cdots + (e_k + f_k) &= (2d_0 - 9e_0) + (2d_1 - 9e_1) + \cdots + (2d_k - 9e_k) \\ &= 2(d_0 + d_1 + \cdots + d_k) - 9(e_0 + e_1 + \cdots + e_k).\end{aligned}$$

If each of these is a multiple of m then so is

$$9(e_0 + e_1 + \cdots + e_k).$$

Now

(a) if $k < m/9$ then $9(e_0 + e_1 + \cdots + e_k)$ is a multiple of m and is less than m, and therefore must be zero;

(b) if $k < m$ and m is not a multiple of 3 then $e_0 + e_1 + \cdots + e_k$ is a multiple of m less than m and again is zero.

Thus in both cases we have

$$e_0 = e_1 = \cdots = e_k = 0$$

and so every digit of n is less than 5.

Q.972

(a) Find all solutions in real numbers of the simultaneous equations

$$x^2 + y^2 = 13\ , \qquad x^3 + y^3 = 35\ .$$

(b) Show that for any given real numbers a and b, the equations

$$x^2 + y^2 = a\ , \qquad x^3 + y^3 = b$$

have at most two real solutions. (Interchanging x and y does not count as a new solution.)

ANS. We solve (b) first. Let

$$x^2 + y^2 = a, \quad x^3 + y^3 = b \tag{1}$$

and write $s = x + y,\ p = xy$. Then

$$\begin{aligned} s^2 &= a + 2p \\ as &= b + ps. \end{aligned}$$

Eliminating p from these equations leads to

$$s^3 - 3as + 2b = 0. \tag{2}$$

If s is a solution of this equation we have

$$p = \frac{1}{2}(s^2 - a)$$

and the equation

$$x^2 - sx + \frac{1}{2}(s^2 - a) = 0 \tag{3}$$

can be solved to find x, and hence y. There will be a real solution if and only if $s^2 - 4 \times \frac{1}{2}(s^2 - a) \geq 0$, that is,

$$s^2 \leq 2a.$$

Now suppose that (2) has the three solutions $s = s_1, s_2, s_3$. Then

$$\begin{aligned} s_1 + s_2 + s_3 &= 0 \\ s_1 s_2 + s_2 s_3 + s_3 s_1 &= -3a \end{aligned}$$

and so

$$s_1^2 + s_2^2 + s_3^2 = (s_1 + s_2 + s_3)^2 - 2(s_1 s_2 + s_2 s_3 + s_3 s_1) = 6a.$$

Therefore we cannot have three different values of s^2 all less than or equal to $2a$. So we have at most two different quadratics (3), leading to at most two solutions to (1).

To solve (a), let $a = 13,\ b = 35$. Then (2) is

$$s^3 - 39s + 70 = 0$$

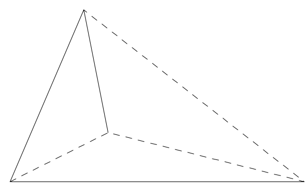
which has solutions $s = 2,\ 5$ and -7. Therefore the quadratics (3) are

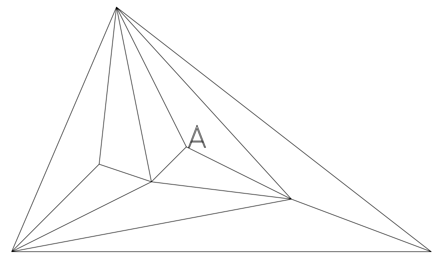
$$\begin{aligned} x^2 - 2x - \frac{9}{2} &= 0, \\ x^2 - 5x + 6 &= 0, \\ x^2 + 7x + 18 &= 0. \end{aligned}$$

The first of these has solutions $x = 1 \pm \frac{1}{2}\sqrt{22}$; the second, $x = 2, 3$; the third, no real solution. Since $y = s - x$ we find that the given equations have solution

$$x = 2,\ y = 3 \quad \text{and} \quad x = 1 + \frac{1}{2}\sqrt{22},\ y = 1 - \frac{1}{2}\sqrt{22}.$$

Q.973 A triangle has n points located inside it, where n is a positive integer. These $n + 3$ points (the three vertices of the triangle and the n interior points) are joined by straight lines in such a way that the original triangle is subdivided into smaller regions, every one of which is a triangle with vertices at three of the given $n + 3$ points. Prove that it is possible to colour the lines red and blue so that for any two points P, Q of the given $n + 3$, there is a blue path from P to Q and also a red path from P to Q.





ANS. The proof will be by mathematical induction. If $n = 1$ the diagram can be coloured as in the left-hand diagram below, where a dotted line denotes blue and a solid line red. Suppose that with n points inside, the triangle can be coloured as required, and consider the case of $n + 1$ points. There will be at least one point (for example A in the right-hand diagram) which is the meeting point of three lines.
Delete these three lines and their common point. By assumption, the remaining diagram can be coloured red and blue so that there is a red path and a blue path from any point to any other. Now reinstate the three lines and colour one of them blue and the other two red. Then the diagram with $n + 1$ interior points also has the required property, and the result has been proved.

Q.974 For any integers x, y an integer $x * y$ is defined by

$$x * y = \min\big(|x+y| \, , \, |x-y| \big) \, .$$

Calculate

(a) $(\cdots((1*2)*3)*\cdots*1995)*1996$;

(b) $1*(2*\cdots*(1994*(1995*1996))\cdots)$;

(c) $(-1996)*((-1995)*\cdots*(0*(1*(2*\cdots*(1994*(1995*1996))\cdots)))\cdots)$.

ANS.

(a) We have

$$\begin{aligned} 1*2 &= 1 \\ (1*2)*3 &= 2 \\ ((1*2)*3)*4 &= 2 \\ (((1*2)*3)*4)*5 &= 3 \end{aligned}$$

and, calculating a few more terms if necessary, it soon becomes clear that

$$(\cdots((1*2)*3)*\cdots*(n-1))*n = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n+1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

The proof by induction is left as an exercise. Hence

$$(\cdots((1*2)*3)*\cdots*1995)*1996 = 998.$$

(b) Here we have

$$\begin{aligned} 1992 * (1993 * (1994 * (1995 * 1996))) &= 1992 * (1993 * (1994 * 1)) \\ &= 1992 * (1993 * 1993) \\ &= 1992 * 0 \\ &= 1992 \end{aligned}$$

and so

$$1 * (2 * \cdots * (1994 * (1995 * 1996)) \cdots) = 1 * (2 * \cdots * (1990 * (1991 * 1992)) \cdots).$$

Continuing in a similar manner we eventually reach

$$\begin{aligned} 1 * (2 * \cdots * (1994 * (1995 * 1996)) \cdots) &= 1 * (2 * (3 * 4)) \\ &= 1 * (2 * 1) \\ &= 1 * 1 \\ &= 0. \end{aligned}$$

(c) First note that from the result of (b) the given expression is

$$(-1996) * ((-1995) * \cdots * (0 * 0)) = (-1996) * ((-1995) * \cdots * (-1 * 0)).$$

Now for any x, y we have

$$\begin{aligned} -x * y &= \min(|-x+y|, |-x-y|) \\ &= \min(|y+x|,\ |y-x|) \\ &= y * x \end{aligned}$$

and so

$$\begin{aligned} &(-1996) * ((-1995) * ((-1994) * \cdots * (-1 * 0))) \\ &= ((-1995) * ((-1994) * \cdots * (-1 * 0))) * 1996 \\ &= (((-1994) * \cdots * (-1 * 0)) * 1995) * 1996 \\ &= (\cdots (0 * 1) * \cdots * 1995) * 1996 \\ &= (\cdots (1 * 2) * \cdots * 1995) * 1996 \\ &= 998. \end{aligned}$$

Q.968 was solved by Daniel Sathianathan, year 12, Epping Boys' High School.

Q.966, 967, 969, 970 and 974 and partial solution to Q.972 were given by Carlos Alberto da Silva Victor, Rio de Janeiro, Brazil.

Q.966, 970, 972 and partial solution to Q.967 were given by Edmond Park (first year, Sydney University).

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