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MATHEMATICS WITH A DIFFERENCE

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In the 3-Unit Maths course you are asked to prove (by induction) various formulae such as

$$1^2 + 2^2 + 3^2 + ... + n^2 = \sum_{x=1}^{n} x^2 = \frac{1}{6}n(n+1)(2n+1),$$

and

$$1^3 + 2^3 + 3^3 + ... + n^3 = \sum_{x=1}^{n} x^3 = \frac{1}{4}n^2(n+1)^2,$$

and

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + ... + \frac{1}{n(n+1)} = \sum_{x=1}^{n} \frac{1}{x(x+1)} = \frac{n}{n+1}.$$

Summation problems like this also turn up in Mathematics competitions and I have even seen some in HSC trial papers for 4-unit maths.
There are a variety of tricks one can use to find sums like this, but here I want to show you a more systematic approach to summation of series using a technique called *'finite calculus'*. You will see shortly why I use the term 'calculus' in this context.
By the way, if you haven't done senior calculus yet, do not despair, you can still follow (most of) the rest of this article.

A function whose domain is the set of integers (or some subset of the integers) is called a *discrete* function. Most of the time we take the domain to be the set of natural numbers.

For example, $f(x) = x^2$ with domain the natural numbers is a discrete function, with $f(0) = 0, f(1) = 1, f(2) = 4$... etc.

We define the **difference** Δf of a discrete function f to be

$$\Delta f = f(x+1) - f(x).$$

(If you have done senior calculus you might like to compare this with the definition of the derivative.)
Here are some examples:

$\Delta x = (x+1) - x = 1.$

$\Delta x^2 = (x+1)^2 - x^2 = 2x + 1.$

$\Delta x^3 = (x+1)^3 - x^3 = 3x^2 + 3x + 1.$

$\Delta \sin(x) = \sin(x+1) - \sin(x) = 2\sin(\frac{1}{2})\cos(x + \frac{1}{2}).$

(using the 'sum to product' formula in the 4-unit maths course.)

$\Delta 2^x = 2^{(x+1)} - 2^x = 2^x.$

(You might like to compare this with $\frac{d}{dx}(e^x) = e^x$!)

As you can see from the above examples, the difference of a polynomial leads to a 'mess'. We would like to find a 'new' sort of polynomial whose difference turns out to be much nicer in some sense. This 'new' polynomial is called a *falling factorial polynomial* and is defined by

$$x^{(n)} = x(x-1)(x-2)...(x-n+1).$$

where n is a positive integer. We pronounce $x^{(n)}$ as 'x falling factorial n'.

(By the way, some modern books write it as $x_{(n)}$, which is probably more logical.)

For example, $x^{(2)} = x(x-1), \quad x^{(3)} = x(x-1)(x-2).$

Note that $x^{(1)} = x$ and we define $x^{(0)}$ to be 1 (as you might expect).

Before seeing why these are so important, observe that we can take any ordinary polynomial and write it in terms of 'falling factorials'.

Eg. $x^2 = x(x-1) + x = x^{(2)} + x^{(1)}.$

$x^3 = x(x-1)(x-2) + 3x^2 - 2x = x^{(3)} + 3x(x-1) + x = x^{(3)} + 3x^{(2)} + x^{(1)}.$

An an exercise you might like to show that

$$x^4 = x^{(4)} + 7x^{(3)} + 6x^{(2)} + x^{(1)}.$$

There is a systematic way of finding the co-efficients using 'Stirling numbers of the second kind', but here we will use a 'bare-hands' approach since we will only be dealing with fairly small powers of x.

We now come to the main reason why these polynomials are so interesting.
Let's compute the difference of some falling factorial polynomials.

$\Delta x^{(1)} = \Delta x = 1$ (as before).

$\Delta x^{(2)} = (x+1)^{(2)} - x^{(2)} = (x+1)x - x(x-1) = 2x = 2x^{(1)}$.

$\Delta x^{(3)} = (x+1)^{(3)} - x^{(3)} = (x+1)x(x-1) - x(x-1)(x-2) = 3x(x-1) = 3x^{(2)}$.

If you've seen basic differential calculus you should now see how this result is very similar to the result about differentiating polynomials!!

In fact, we can show by induction (on n) that

$$\Delta x^{(n)} = nx^{(n-1)}$$

where n is a positive integer. Also Δ can be applied term by term.

Example: $\Delta(x^{(3)} + 2x^{(2)} + 5x^{(1)} + 2) = 3x^{(2)} + 4x^{(1)} + 5$.

Finite Integration:

As with ordinary calculus, we can reverse the differencing process to get a 'finite integral'. For example, from $\Delta x^{(3)} = 3x^{(2)}$, we can write $\Delta^{-1} x^{(2)} = \frac{1}{3}x^{(3)} + C$ where C is a constant. (In fact C is more generally a periodic function of period 1, but we won't worry about this here.)

The symbol Δ^{-1} is called the 'finite integral operator' and corresponds to the integral sign $\int$ in ordinary calculus. Thus, if

$$\Delta f(x) = g(x) \;\text{ then }\; \Delta^{-1} g(x) = f(x) + C.$$

It is not hard to see that $\Delta^{-1} x^{(n)} = \dfrac{x^{(n+1)}}{(n+1)} + C$ for any positive integer n. (You can check this by differencing the right-hand side.)

Having introduced these ideas of the difference and the finite integral of a discrete function, we come now to see how we can use them to evaluate series.

Summation of Series:

Suppose $\Delta f(x) = g(x)$ so that $\Delta^{-1} g(x) = f(x)$ (ignoring the constant C). Remember that the first statement means:

$$f(x+1) - f(x) = g(x).$$

Now substitute $x = 0, x = 1, x = 2, ..., x = n$ into both sides and write down the equations:

$$\begin{gathered}g(0) = f(1) - f(0)\\ g(1) = f(2) - f(1)\\ g(2) = f(3) - f(2)\\ \dots\dots\dots\dots\\ \dots\dots\dots\dots\\ g(n-1) = f(n) - f(n-1)\\ g(n) = f(n+1) - f(n)\end{gathered}$$

NOW ADD!!!

The left-hand side gives $g(0)+g(1)+g(2)+.....+g(n)$ which can be written as $\sum_{x=0}^{n} g(x)$, while on the right-hand side, the terms cancel to give simply $f(n+1)-f(0)$. That is,

$$\sum_{x=0}^{n} g(x) = f(n+1) - f(0) = \Delta^{-1} g(x)\ ]_{x=0}^{x=n+1}.$$

In other words, to sum a series, $\sum_{x=0}^{n} g(x)$, we try to find $\Delta^{-1}g(x)$, the finite integral of $g(x)$, substitute $x=n+1$ and $x=0$ and subtract. More generally, for a sum from $x=a$ to $x=n$, we substitute $x=n+1$ and $x=a$ in the finite integral and subtract.

If you've done some calculus, you will see that just as integration is used to find area, so finite integration is used to do summation. Now for some examples:

Example: Find $\sum_{x=1}^{n} x^2 = 1^2 + 2^2 + 3^2 + ... + n^2$.

Here $g(x) = x^2$ which we must write in terms of falling factorials as $x^{(2)} + x^{(1)}$.

$$\text{So } \sum_{x=1}^{n} x^2 = \Delta^{-1}[x^{(2)} + x^{(1)}] = \frac{x^{(3)}}{3} + \frac{x^{(2)}}{2}\ ]_1^{n+1} = \frac{(n+1)^{(3)}}{3} + \frac{(n+1)^{(2)}}{2} - 0$$

$$= \frac{(n+1)n(n-1)}{3} + \frac{(n+1)n}{2} = \frac{1}{6}n(n+1)(2n+1).$$

as I said at the beginning.

If you would like to try out the above method, you can show that
$\sum_{x=1}^{n} x^3 = \frac{1}{4}n^2(n+1)^2$.

Example: Sum the series $S = 1\times 3 + 2\times 4 + ... + n(n+2)$.

Here $g(x) = x(x+2) = x(x-1) + 3x = x^{(2)} + 3x^{(1)}$.

$$\text{Thus, } S = \Delta^{-1}(x^{(2)} + 3x^{(1)})\ ]_1^{n+1} = \frac{x^{(3)}}{3} + \frac{3x^{(2)}}{2}\ ]_1^{n+1}$$

$$= \frac{1}{6}n(n+1)(2n+7)$$

after simplifying.

As with ordinary integrals, it is convenient to have a table of finite integrals to which we can refer for more complicated functions.

1. $\Delta^{-1}x^{(n)} = \dfrac{x^{(n+1)}}{(n+1)}, \ \ (n \geq 0).$

2. $\Delta^{-1}a^x = \dfrac{a^x}{(a-1)}, \ \ (a \neq 1).$

3. $\Delta^{-1}(a+bx)^{(n)} = \dfrac{(a+bx)^{(n+1)}}{b(n+1)}, \ \ (n \geq 0).$

4. $\Delta^{-1}\cos(a+bx) = \dfrac{1}{2\sin(\frac{b}{2})}\sin(a - \dfrac{b}{2} + bx).$

5. $\Delta^{-1}\sin(a+bx) = \dfrac{-1}{2\sin(\frac{b}{2})}\cos(a - \dfrac{b}{2} + bx).$

Here is an example of a sum involving trigonometric functions.

Example: Find the sum $S = \sin(1) + \sin(2) + \sin(3) + ... + \sin(N)$ (in radians).

$$S = \sum_{x=1}^{x=N} \sin x = \Delta^{-1}\sin x\ ]_1^{N+1} = \frac{-1}{2\sin\left(\frac{1}{2}\right)}\cos\left(x - \frac{1}{2}\right)\ ]_1^{N+1}$$

from the table (5), putting, $a = 0, b = 1$.

$$\text{Thus, } \ S = \frac{1}{2\sin\left(\frac{1}{2}\right)}\left[\cos\left(\frac{1}{2}\right) - \cos\left(N + \frac{1}{2}\right)\right] = \frac{\sin\left(\frac{N+1}{2}\right)\sin\left(\frac{N}{2}\right)}{\sin\left(\frac{1}{2}\right)},$$

after some simplification.

Rising Factorials:

The finite integral result $\Delta^{-1}x^{(n)} = \dfrac{x^{(n+1)}}{(n+1)}$ only works for $n \geq 0$. We want to extend it somehow to accomodate negative values of the integer n. You know (perhaps) from calculus that $\dfrac{d}{dx}\left(\dfrac{1}{x^n}\right) = \dfrac{-n}{x^{n+1}}$ and so $\displaystyle\int \frac{1}{x^n}\,dx = \frac{1}{(1-n)x^{n-1}} + C$ when $n > 1$, and we would like some analogous result for finite integrals. To achieve this, we introduce the notion of 'rising factorials' which are defined as follows.

$$x^{|n|} = x(x+1)(x+2)...(x+n-1), \qquad n \geq 0.$$

This is read as 'x rising factorial n'.

For example, $x^{|3|} = x(x+1)(x+2)$.

It is not now all that difficult to prove by induction that

$$\Delta \frac{1}{x^{|n|}} = \frac{-n}{x^{|n+1|}}, \quad \text{for } n \geq 0.$$

So, for example, $\Delta \frac{1}{x^{|3|}} = \frac{-3}{x^{|2|}}$.

Reversing the process to get a finite integral, we have

$$\Delta^{-1} \frac{1}{x^{|n|}} = \frac{1}{(1-n)x^{|n-1|}} \ \text{ for } n > 1.$$

Thus, for example, $\Delta^{-1} \frac{1}{x^{|2|}} = \frac{1}{-x^{|1|}} = \frac{-1}{x}$.

We can now do a broader range of summation problems.

Example: Sum the series $S = \frac{1}{1 \times 2} + \frac{1}{2 \times 3} + ... + \frac{1}{n(n+1)} = \sum_{x=1}^{n} \frac{1}{x(x+1)}$.

$$= \Delta^{-1} \frac{1}{x(x+1)} \Big]_1^{n+1} = \Delta^{-1} \frac{1}{x^{|2|}} \Big]_1^{n+1} = \frac{-1}{x} \Big]_1^{n+1} = \frac{-1}{n+1} + 1 = \frac{n}{n+1}.$$

Finite Integration by Parts:

A very powerful technique in ordinary integration which is covered in the 4-unit course is Integration by Parts, which gives a method of integrating the product of two functions (if possible). There is a similar technique in finite integration and it is derived from a kind of product rule for differences.
Firstly observe that

$$\Delta f(x)g(x) = f(x+1)g(x+1) - f(x)g(x)$$

$$= f(x+1)g(x+1) - f(x)g(x+1) + f(x)g(x+1) - f(x)g(x)$$
$$= g(x+1)[f(x+1) - f(x)] + f(x)[g(x+1) - g(x)]$$
$$= g(x+1)\Delta f(x) + f(x)\Delta g(x).$$

This the product rule for differences. Rearranging and taking a finite integral we have

$$\Delta^{-1}\left[f(x)\Delta g(x)\right] = f(x)g(x) - \Delta^{-1}\left[g(x+1)\Delta f(x)\right].$$

This is the finite integration by parts formula, and it enables us to do an even wider range of summation problems.

Example: Find $\Delta^{-1}x2^x$.

Put $f(x) = x$ (since it's difference is simple) and $\Delta g(x) = 2^x$ (since its finite integral is easy to find. Then

$$\Delta^{-1}x2^x = x2^x - \Delta^{-1}2^{x+1}.1 = x2^x - 2^{x+1} = 2^x(x-2).$$

We can now use this to find the sum

$T = 1 \times 2^1 + 2 \times 2^2 + 3 \times 2^3 + ... + n2^n.$

$$T = \Delta^{-1}x2^x\ \Big]_1^{n+1} = 2^x(x-2)\ \Big]_1^{n+1} = 2^{n+1}(n-1) + 2.$$

You should now be able to tackle a whole range of summation problems.

In the formula for the finite integral of $\dfrac{1}{x^{|n|}}$ we said that n could not equal 1. It may have crossed your mind at that stage that a similar problem happens in normal integration, that is, we cannot apply the usual formula of calculus to $\displaystyle\int \frac{1}{x}\,dx$. To evaluate this integral we need to introduce the logarithm function, and $\displaystyle\int \frac{1}{x}\,dx = \log x$ (to the base e).

In a similar way, there is no simple way of finding $\Delta^{-1}\dfrac{1}{x}$ or equivalently, there is no simple way to express the sum

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n},$$

as a simple formula.

In the same way as we needed a 'new function', the logarithm, in order to write down the integral of $\dfrac{1}{x}$, so we can introduce the function $\psi(x)$ (read as 'psi of x') in order to write down a 'closed formula' for the above sum. In fact we write $\Delta^{-1}\dfrac{1}{x} = \psi(x)$ and so

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} = \psi(n+1) - \psi(1).$$

This function has many interesting properties, including $\displaystyle\int_x^{x+1} \psi(t)\,dt = \log x$.

Finite Calculus has many applications apart from evaluating series. It can be used in curve fitting, error anaylsis, and in solving equations called difference equations, which often turn up when one is looking at so-called discrete problems.

You might like to make up some series problems of your own and try to solve them using the method of difference calculus. Here are two for you to try,

1) $$\frac{1\times 2}{3}+\frac{2\times 3}{3^2}+\frac{3\times 4}{3^3}+....+\frac{n(n+1)}{3^n}.$$

2) $$\sum_{x=2}^{n}\frac{1}{x^2-1}.$$