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CURVE SKETCHING - WITH A DIFFERENCE

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When you began to sketch curves early in high school, you evaluated the "y-value" for several "x-values", plotted the resulting points and then joined them up as smoothly as you could. The second half of this process is actually very close to what a scientist does with an experiment: given a set of readings $y_1, y_2, \cdots y_n$ for an apparatus with settings $x_1, x_2, \cdots x_n$, find the function $y = f(x)$ whose graph passes through the points $(x_1, y_1),\ (x_2, y_2), \cdots, (x_n, y_n)$. For example suppose we know that the points (x, y) lie on the graph of some polynomial where

$$\begin{array}{rcrrrrrrr} x & = & -3 & -2 & -1 & 0 & 1 & 2 & 3 \\ \\ y & = & 24 & 16 & 10 & 6 & 4 & 4 & 6 \end{array}$$

How do we find the equation of this graph?

Questions like this can be answered using an idea introduced by Peter Brown in his Finite Calculus article on page 7. Suppose the polynomial we are looking for is

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \tag{1}$$

where we do not know the coefficients $a_0, a_1, \cdots a_n$ – or even the degree n. As Peter Brown did, we will introduce the difference of a function $\Delta f(x) = f(x+1) - f(x)$ and the falling factorial polynomial:

$$x^{(r)} = x(x-1)\cdots(x-r+1)$$

where $\Delta x^{(r)} = rx^{(r-1)}$. The polynomial (1) can be re-written as

$$f(x) = b_n x^{(n)} + b_{n-1}x^{(n-1)} + \cdots + b_1 x^{(1)} + b_0. \tag{1'}$$

Thus

$$\begin{aligned} \Delta f(x) &= b_n \Delta x^{(n)} + b_{n-1}\Delta x^{(n-1)} + \cdots + b_2 \Delta x^{(2)} + b_1 \Delta x^{(1)} \\ &= nb_n x^{(n-1)} + (n-1)b_{n-1}x^{(n-2)} + \cdots + b_2 x^{(1)} + b_1. \end{aligned}$$

Now this difference can be applied again where, just as you learned to write the second derivative as $\dfrac{d^2y}{dx^2}$, so we will write the *second difference* as

$$\Delta^2 f(x) = \Delta(\Delta f(x)) = n(n-1)b_n x^{(x-2)} + (n-1)(n-2)b_{n-1}x^{(n-3)} + \cdots + 2b_2.$$

Similarly, writing $\Delta^3 f(x)$ for $\Delta(\Delta(\Delta f(x)))$ etc, we have

$$\begin{aligned}
\Delta^3 f(x) &= n(n-1)(n-2)b_n x^{(n-3)} + (n-1)(n-2)(n-3)b_{n-1}x^{(n-4)} + \cdots \\
\Delta^n f(x) &= n! \times b_n \\
\Delta^{n+1} f(x) &= 0.
\end{aligned}$$

This gives the rule:

The degree of the polynomial $f(x)$ is the smallest number n where $\Delta^{n+1} f(x) = 0$.

If we apply $\Delta, \Delta^2, \cdots$ to our data, we get

$$\begin{array}{rcrrrrrrr}
x & = & -3 & -2 & -1 & 0 & 1 & 2 & 3 \\
y = f(x) & = & 24 & 16 & 10 & 6 & 4 & 4 & 6 \\
\Delta y = \Delta f(x) & = & -8 & -6 & -4 & -2 & 0 & 2 & \\
\Delta^2 y = \Delta^2 f(x) & = & 2 & 2 & 2 & 2 & 2 & &
\end{array}$$

So, since $\Delta^3 y$ is obviously 0, $f(x)$ is a quadratic, and so

$$\begin{aligned}
f(x) &= b_2 x^{(2)} + b_1 x^{(1)} + b_0 \\
\Delta f(x) &= 2b_2 x^{(1)} + b_1 \\
\Delta^2 f(x) &= 2b_2
\end{aligned}$$

From the table, we see that $\Delta^2 y = 2$ and so, from this last equation, $b_2 = 1$. Now if we substitute $x = 0$ into the second last equation, we get

$$\begin{array}{lrcl}
 & b_1 & = & \Delta f(0) = -2 \quad \text{(from the table)} \\
\text{similarly,} & b_0 & = & f(0) = 6 \\
\text{so} & f(x) & = & x(x-1) - 2x + 6 \\
 & & = & x^2 - 3x + 6
\end{array}$$

Example 2: Consider the data

$$\begin{array}{rcrrrrrrr}
x & = & -3 & -2 & -1 & 0 & 1 & 2 & 3 \\
y & = & 60 & 10 & 4 & 6 & 4 & 10 & 60 \\
\Delta y & = & -50 & -6 & 2 & -2 & 6 & 50 & \\
\Delta^2 y & = & 44 & 8 & -4 & 8 & 44 & & \\
\Delta^3 y & = & -36 & -12 & 12 & 36 & & & \\
\Delta^4 y & = & 24 & 24 & 24 & & & &
\end{array}$$

So the points (x, y) lie on a curve which represents a polynomial $f(x)$ of degree 4 :

$$\begin{aligned}
f(x) &= b_4 x^{(4)} + b_3 x^{(3)} + b_2 x^{(2)} + b_1 x^{(1)} + b_0 \\
\Delta f(x) &= 4b_4 x^{(3)} + 3b_3 x^{(2)} + 2b_2 x^{(1)} + b_1 \\
\Delta^2 f(x) &= 12b_4 x^{(2)} + 6b_3 x^{(1)} + 2b_2 \\
\Delta^3 f(x) &= 24b_4 x^{(1)} + 6b_3 \\
\Delta^4 f(x) &= 24b_4
\end{aligned}$$

From the table,

$$\begin{array}{rcrcrll} 24b_4 &=& \Delta^4 y(0) &=& 24 & \text{and so} & b_4 = 1 \\ 6b_3 &=& \Delta^3 y(0) &=& 36 & \text{and so} & b_3 = 6 \\ 2b_2 &=& \Delta^2 y(0) &=& 8 & \text{and so} & b_2 = 4 \\ b_1 &=& \Delta y(0) &=& -2 & & \\ b_0 &=& y(0) &=& 6 & & \end{array}$$

so
$$\begin{aligned} y &= x^{(4)} + 6x^{(3)} + 4x^{(2)} - 2x + 6 \\ &= x(x-1)(x-2)(x-3) + 6x(x-1)(x-2) + 4x(x-1) - 2x + 6 \\ &= x^4 - 3x^2 + 6 \end{aligned}$$

Example 3: Consider the readings:

$x =$	0	1	2	3	4	5
$y =$	4	12	36	108	324	972
$\Delta y =$	8	24	72	216	648	
$\Delta^2 y =$	16	48	144	432		

This is not looking very hopeful! However notice that each of the difference lines is just twice the line above, e.g. $\Delta y = 2y$.

Now, looking at the table of differences in Peter Brown's article, we see that, if $y = f(x)$ where $f(x) = Aa^x$, then

$$\Delta y = \Delta(Aa^x) = A(a-1)a^x = (a-1)y$$

which is the result we found (with $a = 3$). So

$$y = A \times 3^x$$

where

$$A = f(0) = 4.$$

Here are some for you to try:

1.

$x =$	-3	-2	-1	0	1	2	3
$y =$	0	0	0	6	24	60	120

2.

$x =$	-3	-2	-1	0	1	2	3
$y =$	408.429	20.0855	2.71828	1	1	2.71828	20.0855

[Hint: First find the values of $\ln(y)$.]

SAFETY FIRST

Replace the letters in the following by different non-zero digits to give a correct sum:

$$\begin{array}{ccccc} & M & A & K & E \\ & S & A & F & E \\ \hline P & L & A & N & S \\ \hline \end{array}$$

Answers are in the solutions section