Parabola Volume 33, Issue 2 (1997)

Fractals: How long is a piece of string?

Bruce Henry and Clio Cresswell

> *"And though the holes were rather small,*
> *they had to count them all.*
> *Now they know how many holes it takes to fill the Albert Hall.*
> *I'd love to turn you on."*
> John Lennon and Paul McCartney 1967

You have probably all heard the expression "How long is a piece of string?". It's usually offered in rhetorical response to a question that has no sensible answer. On the other hand if somebody was to give you a particular piece of string and then ask the question "How long is this piece of string?", you would find no problem in coming up with a numerical answer. For a mathematician the rhetorical question "How long is a piece of string?" is a legitimate question even in the absence of a particular piece of string. To find the length L of a piece of string (or any curve for that matter) take a known length l and count the number of times n that this length would need to be laid end to end in order to pass from the beginning to the end of the curve. The length of the curve is then $L \approx nl$. We will refer to this method for measuring the length as the divider's method. If the curve is in the shape of a straight line then the divider's method will yield a result that differs from the exact result by a distance less than l. In general the error in the length estimate obtained using the divider's method will depend on the shape of the curve; however in general if we repeat our measurement process for different l becoming smaller and smaller we might expect this error to become smaller and smaller. But what if we continue to apply the divider's method down to where l approaches zero? Surely n must now approach infinity and our length will be approximated by the product $0 \times \infty$ which is not defined since ∞ is not a number. In order to continue to apply the divider's method all the way down to where l approaches zero we need to employ mathematical limits. We use the notation

$$\lim_{l \to 0}$$

to represent l approaching zero and

$$\lim_{n \to \infty}$$

to represent n approaching infinity. We may now write the formula for the length of the curve or the length of a piece of string as

$$L = \lim_{l \to 0} \lim_{n \to \infty} nl.$$

We could also write the length formula as

$$L = \lim_{l \to 0} n(l)l$$

where we note that n is a function of l.

The crucial assumption underlying the divider's method is that curves appear increasingly smoother (more like straight line segments) under magnification. As an illustration consider the perimeter of a circle. Figure 1 shows such a perimeter with a small piece magnified. When only a very small segment of the perimeter is visible to us it looks like a straight line.

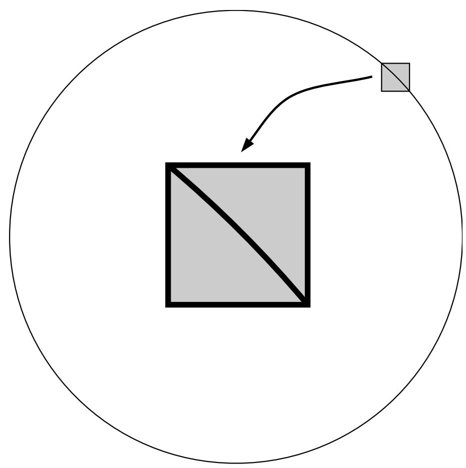

Figure 1

Figure 2 shows a circle whose perimeter is approximated by $n = 12$ straight line segments each of length l. Each straight line segment subtends an angle θ. The true length of the perimeter is simply $L = ns$ where s is the arc length that subtends the angle θ. Since the sum of the angles must add up to 2π we have $n = 2\pi/\theta$ and the arc length is $s = r\theta$ so that the true length of the perimeter is $L = 2\pi r$. Now consider the divider's method. The geometry is shown in Figure 2 where s is the arc length between points B and D and l is the straight line distance between these points.

Using simple trigonometry applied to triangle ABC we have

$$\sin\left(\frac{\theta}{2}\right) = \frac{l}{2r}.$$

This may be rearranged to give

$$l = 2r\sin\left(\frac{\theta}{2}\right) = \text{the length of each straight line segment}$$

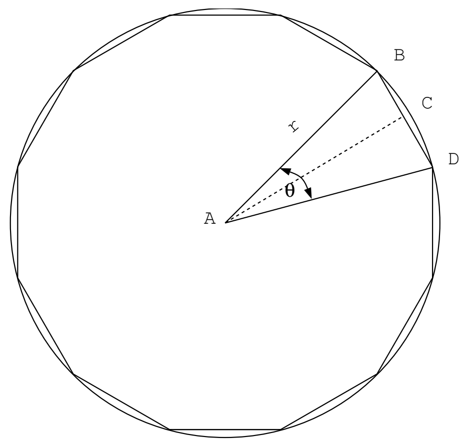

Figure 2

or

$$\theta = 2\sin^{-1}\left(\frac{l}{2r}\right) = \text{the angle subtended by each straight line segment.}$$

This implies that

$$L \approx nl = \frac{4\pi r}{\theta}\sin\left(\frac{\theta}{2}\right) \quad \text{with } L \text{ expressed in terms of } \theta$$

or

$$L \approx \frac{\pi l}{\sin^{-1}\left(\frac{l}{2r}\right)} \quad \text{with } L \text{ expressed in terms of } l.$$

Clearly the approximation for L will be good provided that the straight line segment l is a good approximation to the arc segment s. As we have seen in Figure 1 and 2 this approximation is very good when s is small relative to the radius. But when s is small relative to the radius, l is small relative to the radius and θ is also small. This allows us to use small angle approximations for the trigonometric function $\sin$. In particular if x is small then $\sin(x) \approx x \approx \sin^{-1}(x)$ (e.g., $\sin(.01) = .09983\ldots$ and $\sin^{-1}(.01) = .01001\ldots$). Using the small angle approximation in the formula $L = nl$ we have

$$L \approx \frac{4\pi r}{\theta}\frac{\theta}{2} = 2\pi r$$

or

$$L \approx \frac{\pi l}{\frac{l}{2r}} = 2\pi r.$$

Now consider the employment of the divider's method for measuring the length of the coastline of mainland Australia. Our analysis for the circle suggests that the divider's method might yield a reasonable result provided that the length of our straight line segment is less than the average 'radius' of Australia. Australia has an area of some $7,617,930$ square kilometres[1] which evaluates to an average radius of about 1557km

[1] Data from the CIA 1996 world factbook; net page: http://www.odci.gov/cia/publications/nsolo/factbook/as.htm

and so we might anticipate reasonable coastline measurements based on straight line segments of a few hundred kilometres or less. Surveys of the coastline of Australia reveal:

Length of line segment l	Number of segments n	Length of coastline $L \approx nl$
500 km	23.9	[2]11,950 km
250 km	52.8	[2]13,200 km
100 km	144.2	[2]14,420 km
1 km	25,760	[3]25,760 km

From the data in the table it is clear that nl does not converge to a constant value as l decreases and so $L \approx nl$ does not provide a reliable approximation to the length of the coastline. The most accurate estimate of the coastline of mainland Australia to date is 35,877 kilometres[4]. In this estimate the coastline is defined as the Mean High Water mark which is approximated by a series of straight line segments of unequal lengths ranging between a minimum of 20 metres and a maximum of 8000 metres[5]. One reported estimate of the length of the Australian Coastline based on high resolution satellite imagery put it at $130,000$ kilometres which is roughly ten times the diameter of the Earth. The reason for the discrepancies between the various measurements and the impossibility to pin it down precisely is that as we zoom in on the coastline of Australia we encounter more and more detail with a roughness on all measurable scales. The Australian coastline cannot be approximated by an n sided polygon. Estimates for the length of the coastline increase as the size of the straight line approximating segments is reduced. The final length of the coastline using the divider's method will depend on the limiting size of the 'smallest' straight line segment, if such a limiting size exists. The smallest length scale known at present is at the size of quarks (about 10^{-16} metres). Dividing the area of Australia by the area occupied by quarks we estimate that Australia consists of about $n = 10^{45}$ of them. Not all of these reside on Australia's perimeter but this gives an upper bound estimate of the coastline at about 10^{26} km. If we travelled out into space in a direct line for this distance then we would be beyond the furthest object ever observed in our universe.

We now consider an imaginary mathematical island called the Koch snowflake. This island has detail on all scales, there is no limiting physical atomic scale. As we shall see this island can be constructed using Turtle Graphics[6] and string substitution rules. Consider an imaginary turtle on your page that can draw images by following simple instructions such as i) move forward a distance d and draw a line, ii) change direction by turning through an angle θ to the right and iii) change direction by turning

[2] L.F. Richardson "The Problem of Contiguity: An appendix to the statistics of deadly quarrels", *General Systems*, **6** (1961) 139.

[3] Data from the CIA 1996 world factbook.

[4] Measurement by The Australian Surveying and Land Information Group (AUSLIG); net page http://www.auslig.gov.au/facts.htm#dimensions.

[5] We are greatful to AUSLIG for providing us with this information

[6] Turtle Graphics was invented by Seymour Papert in the late nineteen sixties at the MIT Artificial Intelligence Laboratories.

through an angle θ to the left. To simplify our code we will label the first of these instructions F, the second instruction $+$, and the third instruction $-$. Suppose we specify the angle $\theta = \frac{2\pi}{6}$. What image would be generated by the string $F - -F - -F$? The answer is an equilateral triangle. What image would be generated by the string $F + F - -F + F$? The answer is shown in Figure 3.

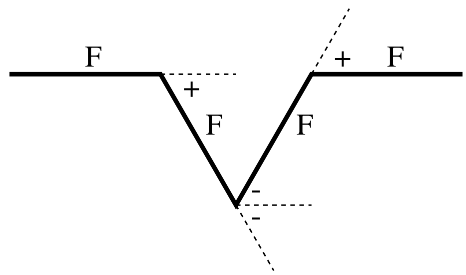

Figure 3

Now suppose we start with the character string for the equilateral triangle

$$\boxed{\text{F}} - -\boxed{\text{F}} - -\boxed{\text{F}}$$

and we replace each $\boxed{\text{F}}$ in this string with the substitution string $\boxed{\text{F+F- -F+F}}$ then we obtain the new string

$$\boxed{\text{F+F- -F+F}} - -\boxed{\text{F+F- -F+F}} - -\boxed{\text{F+F- -F+F}}.$$

Ignoring the boxes, which were used simply to highlight the substitution process, can you draw the image corresponding to this character string? We have really replaced each straight line in the equilateral triangle by the image shown in Figure 3. The result of this replacement is shown in Figure 4. Here we have also reduced the original length scale d. Now in this new string replace each F again by the substitution string $F + F -$

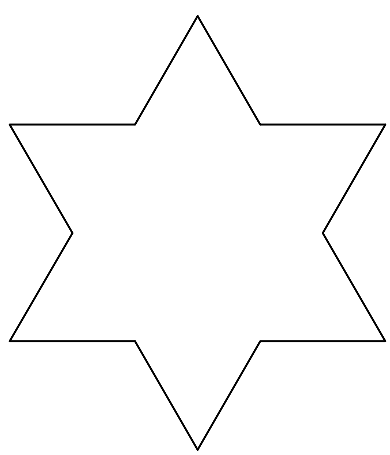

Figure 4

$-F+F$. This now yields the new string $F+F--F+F+F+F--F+F--F+F--F+$

$F{+}F{+}F{-}{-}F{+}F{-}{-}F{+}F{-}{-}F{+}F{+}F{+}F{-}{-}F{+}F{-}{-}F{+}F{-}{-}F{+}F{+}F{+}F{-}{-}F{+}F{-}$
${-}F{+}F{-}{-}F{+}F{+}F{+}F{-}{-}F{+}F{-}{-}F{+}F{-}{-}F{+}F{+}F{+}F{-}{-}F{+}F$. The corresponding image is shown in Figure 5 where we have reduced the length scale by a factor of three. The true Koch snowflake is the image obtained from the resultant character string after

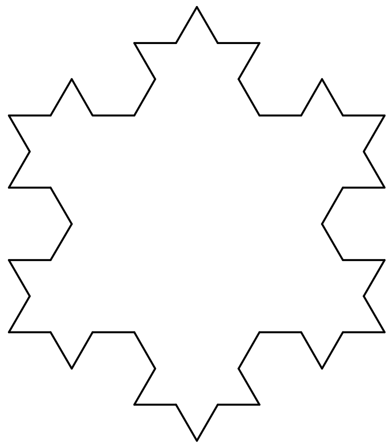

Figure 5

the substitution string has been applied an infinite number of times. Each time we apply the substitution string we also reduce the length scale by a further factor of three [7]. Figure 6 shows the image obtained after five substitutions. What is the length

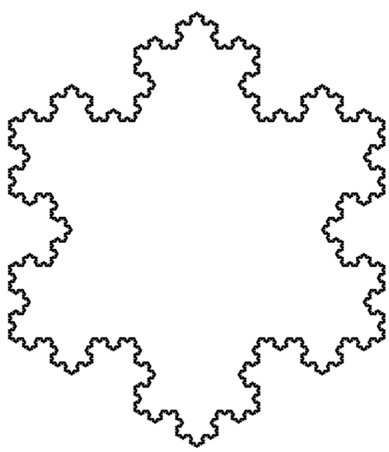

Figure 6

of the perimeter of the Koch snowflake? The perimeter is the number of F characters multiplied by the corresponding length scale. In the Koch snowflake there are infinitely many F characters but the length scale has been reduced by a factor three an infinite number of times so again we have a result like $L = 0 \times \infty$ and again we evaluate

[7] The starting string together with the substitution string and the scale factor is a convenient code for the image. Codes like this, which were first employed by Aristid Lindenmayer to model biological growth, are called L-systems.

the expression for L by properly taking limits. Starting with the equilateral triangle, the number of F characters after n substitutions is 3×4^n. The length scale after n substitutions is $d/3^n$. Hence the perimeter after n substitutions is

$$L(n) = 3l\frac{4^n}{3^n}.$$

After an infinite number of substitutions, ie. as $n \to \infty$ we have[8]

$$L = \lim_{n\to\infty} 3l\left(\frac{4}{3}\right)^n \to \infty.$$

On the other hand the area of the Koch snowflake is finite. We can evaluate the area as follows. The area of the equilateral triangle of side d is $d^2\sqrt{3}/4$. After one substitution there are four equilateral triangles, one of side d and three of side $d/3$ giving an overall area $(d^2\sqrt{3}/4)(1 + 3(1/3^2))$. After n substitutions the overall area is

$$\begin{aligned} A(n) &= \left(\frac{d^2\sqrt{3}}{4}\right)\left(1 + 3\sum_{j=1}^{n}\frac{4^{j-1}}{3^{2j}}\right) \\ &= \left(\frac{d^2\sqrt{3}}{4}\right)\left(1 + \frac{3}{4}\sum_{j=1}^{n}\left(\frac{4}{9}\right)^j\right) \end{aligned}$$

Evaluating the sum[9] we have

$$A(n) = \left(\frac{d^2\sqrt{3}}{4}\right)\left(1 + \frac{3}{4}\left(\frac{1-(\frac{4}{9})^n}{1-\frac{4}{9}}\right)\right)$$

Hence after an infinite number of substitutions we have

$$\begin{aligned} A &= \lim_{n\to\infty} A(n) \\ &= \left(\frac{d^2\sqrt{3}}{4}\right)\left(1 + \frac{3}{4}\left(\frac{1}{1-\frac{4}{9}}\right)\right) \\ &= 47\sqrt{3}d^2/80 \end{aligned}$$

[8] If n is a positive integer and a is a fixed positive constant less than one then

$$\begin{aligned} \lim_{n\to\infty} a^n &\to 0 \\ \lim_{n\to\infty} 1/a^n &\to \infty \end{aligned}$$

[9] Here we use the geometric series

$$\sum_{j=1}^{n} a^j = \frac{1-a^n}{1-a}.$$

Our investigations into the length of a curve have revealed that the length depends fundamentally on the shape of the curve. If the shape can be approximated by a series of straight line segments (even if vanishingly small) then the length is well defined. This was found to be the case for the perimeter of a circle. A circle has a finite area enclosed by a finite length perimeter. On the other hand the perimeter of the Koch snowflake cannot be approximated by a polygon. The Koch snowflake has a finite area enclosed by an infinite length perimeter. Australia has a finite area enclosed by a perimeter whose length increases as the length scale of the measuring device decreases until the limiting size of the measuring device is reached. Shapes with a finite area but an infinite length perimeter or a perimeter whose length increases as the length scale of the measuring device decreases cannot be described by standard geometry. In standard Euclidean geometry a perimeter enclosing a finite two-dimensional region must have a finite length. Indeed any one-dimensional line enclosing a finite two-dimensional area must have a finite length. In Euclidean geometry we learn that points have zero dimension, lines have dimension one, planes have dimension two etc. The coastline of Australia and the Koch snowflake are lines but they cannot be one-dimensional. What is the dimension of the coastline of Australia or the Koch snowflake if it is not one?

There are many different ways to measure the dimension of shapes. Here we will consider the simplest of these measurements, the Kolmogorov Capacity. The idea is to cover the shape of interest with small cells [10] of size ϵ and then to count the number of cells $M(\epsilon)$ that contain some part of the shape. The dimension is then defined as[11]

$$D = \lim_{\epsilon \to 0} \frac{\log\left(M(\epsilon)\right)}{\log(\frac{1}{\epsilon})}.$$

For shapes in nature we cannot really take the limit $\epsilon \to 0$ and so we measure $M(\epsilon)$ for a number of different but small values of ϵ and then we plot the points $(\log(1/\epsilon), \log(M(\epsilon)))$. If these points fit on a straight line $y = mx$ where $y = \log(M(\epsilon))$ and $x = \log(1/\epsilon)$ then the slope of the line is

$$m = \frac{\log\left(M(\epsilon)\right)}{\log(\frac{1}{\epsilon})}$$

which is the same as our expression for the fractal dimension - provided that our ϵ values are sufficiently small. Figure 7 shows isolated points covered by cells of size ϵ. If there are n isolated points then we expect that for sufficiently small ϵ these points should be in different cells and hence $M(\epsilon) = n$. The dimension of the set of isolated

[10]These cells can be straight line segments, squares, circles, triangles, cubes etc.

[11]The logarithm function used in the expression for the dimension has the following properties:

$$\begin{aligned} \log(1) &= 0 \\ \log(xy) &= \log(x) + \log(y) \\ \log(x/y) &= \log(x) - \log(y) \\ log(x^n) &= n\log(x) \end{aligned}$$

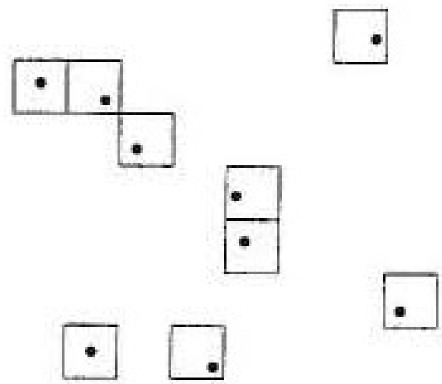

Figure 7

points is then[12]

$$D \approx \lim_{\epsilon \to 0} \frac{\log n}{-\log \epsilon} \to 0.$$

Figure 8 shows a smooth line segment covered by cells of size ϵ. If the total length of

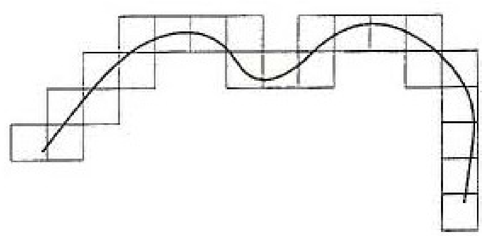

Figure 8

the line is L then we expect that the number of cells containing part of the line is the

[12]If x is a positive variable and a is a fixed positive constant then

$$\begin{aligned}
\lim_{x\to 0} a/x &\to \infty \\
\lim_{x\to 0} ax &\to 0 \\
\lim_{x\to 0} x/x &\to 1 \\
\lim_{x\to\infty} a/x &\to 0 \\
\lim_{x\to\infty} a/\log(x) &\to 0 \\
\lim_{x\to 0} a/\log(x) &\to 0
\end{aligned}$$

length of the line divided by the size of the cell, i.e., $M(\epsilon) = L/\epsilon$. The dimension of the smooth line is thus

$$
\begin{aligned}
D &\approx \lim_{\epsilon \to 0} \frac{\log\left(\frac{L}{\epsilon}\right)}{-\log \epsilon} \\
&= \lim_{\epsilon \to 0} \frac{\log L - \log \epsilon}{-\log \epsilon} \to 1
\end{aligned}
$$

Figure 9 shows an area bounded by a smooth perimeter covered by cells of size ϵ. If

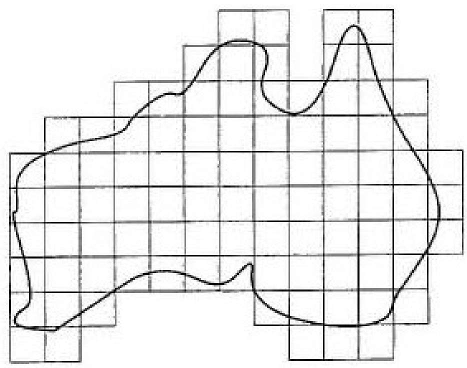

Figure 9

the total area is A then we expect that the number of cells containing part of this area is the total area divided by the area of a cell, i.e., $M(\epsilon) = A/\epsilon^2$. The dimension of the area is thus

$$
\begin{aligned}
D &\approx \lim_{\epsilon \to 0} \frac{\log\left(\frac{A}{\epsilon^2}\right)}{-\log \epsilon} \\
&= \lim_{\epsilon \to 0} \frac{\log A - 2\log \epsilon}{-\log \epsilon} \to 2
\end{aligned}
$$

Thus isolated points have dimension zero, smooth lines have dimension one and regions bounded by smooth lines have dimension two. Now consider the perimeter of the Koch snowflake. Here we can take our cells to be straight line segments of length ϵ. From above we have that for $\epsilon = d/3^n$ the number of segments needed to cover the perimeter is $M(\epsilon) = 3 \times 4^n$ and hence

$$
\begin{aligned}
D &= \lim_{\epsilon \to 0} \frac{\log(M(\epsilon))}{-\log \epsilon} \\
&= \lim_{n \to \infty} \frac{\log(3 \times 4^n)}{-\log(d/3^n)} \\
&= \lim_{n \to \infty} \frac{\log 3 + n \log 4}{n \log 3 - \log d} \\
&= \frac{\log 4}{\log 3}
\end{aligned}
$$

Hence the dimension of the perimeter of the Koch snowflake is about 1.26. The dimension greater than one but less than two tells us that the line is not smooth but has wiggles on wiggles that give it a thickness but not enough thickness to have a finite area. Shapes that have a non-integer number (or fractional number) of dimensions are called *fractals*. Dimension measurements that allow for the possibility of non-integer dimensions are called *fractal dimensions.* As an historical footnote, the word *fractal* was introduced by Benoit Mandelbrot in 1975 who was the first to realize that many shapes in nature exhibit a fractal structure. The construction of mathematical curves whose length could not be measured (so called nonrectifiable curves) dates back to Peano (1890)[13] and Koch (1904). Dimension measurements that allow for the possibility of non-integer dimensions date back to Minkowski (1901) and Hausdorrf (1919).

As an example of a fractal shape in nature we return to consider the coastline of mainland Australia. Figure 10 shows a plot of the points $(\log(1/\epsilon), \log(M(\epsilon)))$ for the four sets of values given in the table above. Here ϵ is the length of the line segment and $M(\epsilon)$ is the number of line segments needed to cover the coastline. The dimension

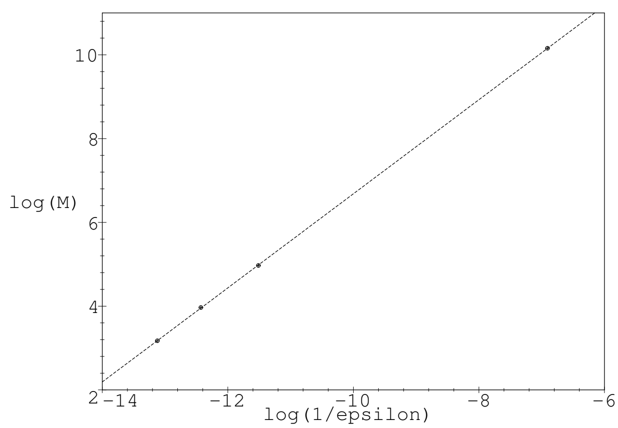

Figure 10

can be estimated from the slope of the straight line of best fit as shown in Figure 10. This yields the dimension estimate 1.12 confirming that the coastline of Australia is a fractal.

In conclusion we present a new formula for the length of a piece of string,

$$L \approx \lim_{l \to 0} a l^{-D+1},$$

where a is a finite constant, l is a known length and D is the fractal dimension. If $D = 1$ then this reduces to $L = a$; however if $D > 1$ then $L \to \infty$ and if $D < 1$ then $L \to 0$. So next time somebody asks you the rhetorical question "How long is a piece of string?" you might like to respond with the rejoinder "That depends, what is its fractal dimension?".

[13] Peano discovered a curve that twists around in such a complicated fashion that it passes through every point inside a two-dimensional region.