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UNSW SCHOOL MATHEMATICS COMPETITION 1997

SOLUTIONS

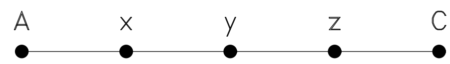
JUNIOR DIVISION

1. Find the smallest positive integer n such that $\frac{1}{3}n$ is a perfect cube, $\frac{1}{5}n$ a perfect fifth power and $\frac{1}{7}n$ a perfect seventh power.

   Solution. Write n as a product of prime numbers
   $$n = 2^a \times 3^b \times 5^c \times 7^d \times 11^e \times \cdots \ .$$
   Then
   $$\tfrac{1}{3}n = 2^a \times 3^{b-1} \times 5^c \times 7^d \times 11^e \times \cdots \ ,$$
   and for this to be a perfect cube all of the exponents $a, b-1, c, d, e, \ldots$ must be multiples of 3. In the same way we want $a, b, c-1, d, e, \ldots$ to be multiples of 5 in order to make $\frac{1}{5}n$ a perfect fifth power, and $a, b, c, d-1, e$ to be multiples of 7 so that $\frac{1}{7}n$ will be a seventh power. First, $a, e, f, \ldots$ must be multiples of $3, 5$ and 7; since we are looking for the smallest possible value of n we shall take $a = e = f = \cdots = 0$. Next, b is a multiple of 5 and of 7, so b is a multiple of 35 and we can say $b = 35t$. But also $b-1$ is a multiple of 3, that is, $35t-1$ is a multiple of 3, and so $2t-1$ is a multiple of 3. Clearly the smallest possible value of t is 2, and so $b = 70$. Similarly (check the details for yourself) we find that the smallest possible values of c and d are 21 and 15 respectively. So the required integer is
   $$n = 3^{70} \times 5^{21} \times 7^{15} \ .$$
   (As this is a 61-digit number we did not expect you to actually calculate it!)

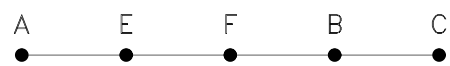
2. On a small island there are six towns A, B, C, D, E and F. The transport system consists of a number of two-way roads between pairs of towns. Roads do not meet each other except at towns. Not all pairs of towns are directly connected; for example, to travel from A to D it is necessary to pass through one of the other towns. The shortest trip from A to C passes through three of the other towns; the shortest trips from A to B, from C to D and from C to E each pass through two other towns. No other trip requires more than one intermediate town. Make a sketch showing which pairs of towns are connected directly by roads.

   Solution. Since the shortest trip from A to C passes through three other towns we may begin by drawing the following map.

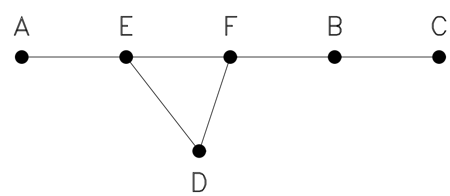


Now clearly, even if we draw more roads later, towns y and z can be reached from C with fewer than two intermediate towns. So y and z cannot be D and E (which require two intermediate towns coming from C) and therefore must be B and F.

Which is which? If z were to be F, then since we can travel from A to F passing through just one other town (or none at all), there would be a trip $A-?-F-C$ or $A-F-C$. But this is impossible as we must pass through at least three other towns in travelling from A to C. So z is B and y is F. Since A and D are not joined directly, this also means that x must be E.



Now there can be no more roads joining A, E, F, B and C, for any such road would give a shorter trip from A to C. The only way to place D so that travelling from A to D requires one intermediate town is to join it to E; and then, in order to get from C to D via two intermediate towns, a road is needed from D to F. So the complete map of the island is as follows.



3. Given two primes p and q, how many pairs (x, y) of integers are there for which

$$\frac{p}{x} + \frac{q}{y} = 1\ ?$$

Solution. Multiplying both sides by xy and rearranging gives

$$xy - xq - yp = 0$$

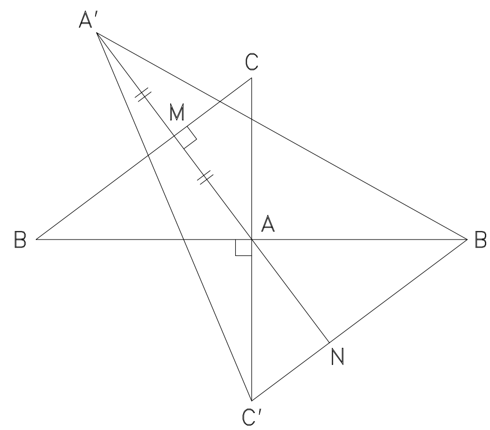
and hence

$$(x-p)(y-q) = xy - xq - yp + pq = pq\ . \tag{$*$}$$

Since p and q are prime, pq has only four factorisations as a product of two positive integers, and another four into negative integers. So there are eight possible pairs (x, y) satisfying equation $(*)$. However one of these pairs has $x = 0, y = 0$ which is obviously impossible in the original problem. So there are seven pairs (x, y) for which the given equation is true.

4. Let $\triangle ABC$ be right-angled. Let A' be the mirror image of the vertex A in the side BC, let B' be the mirror image of B in AC and C' the mirror image of C in AB. Find the ratio

$$\text{area}(\triangle ABC)/\text{area}(\triangle A'B'C') \ .$$



Solution. Consider the diagram on the right. Clearly $\triangle ABC$ and $\triangle A'B'C'$ are congruent, and so $B'C'$ is parallel to BC. Also $\triangle ACM$ and $\triangle AC'N$ are congruent; hence AN is perpendicular to $B'C'$ and is equal in length to AM. Therefore $A'MAN$ is a straight line, is an altitude of $\triangle A'B'C'$, and is three times as long as AM.

This shows that $\triangle ABC$ and $\triangle A'B'C'$ stand on equal bases BC and $B'C'$, and the former has altitude one third of the latter; so the ratio of areas is

$$\frac{\text{area}(\triangle ABC)}{\text{area}(\triangle A'B'C')} = \frac{1}{3} \ .$$

5. The students in a school are arranged in a circle and numbered $1, 2, 3, \ldots$. Starting with number 2, every second student is sent away to clean up part of the schoolyard, until only one student is left, and is allowed to go home early. (So, if there were eleven students, they would be chosen in the order $2, 4, 6, 8, 10, 1, 5, 9, 3, 11$ and the student to go home early would be number 7.)

   (a)If there are 1997 students in the school, which one gets to go home early?

   (b)How many students may there be in the school if number 1997 is to be the one who goes home early?

Solution. If there are n students altogether, let $h(n)$ be the one who goes home early. (So the example above shows that $h(11) = 7$.) Suppose that n is even, $n = 2m$. Then on the first turn around the circle, the students numbered $2, 4, 6, \ldots, 2m$ are sent off to clean up the yard. The remaining m students are in exactly the same position as they would have been if we had started with m students, except that the first student is number 1, the second is number 3, the third number 5, … the kth number $2k - 1$. Now the student who goes home is the one in position $h(m)$ among these m students, that is, number $2h(m) - 1$ in the original $2m$ students. Hence

$$h(2m) = 2h(m) - 1 \ .$$

(For example, from above we can immediately say that out of 22 students, number 13 would go home early.) On the other hand, suppose that n is odd, $n =$

$2m+1$. Then the first $m+1$ students to be sent on clean-up duty are numbers $2, 4, 6, \ldots, 2m$ and 1; this leaves m students in a similar position to those above, except that the first is number 3, the second number 5, … and the kth number $2k+1$. So the student dismissed early is the one in position $h(m)$ from this list, that is, number $2h(m)+1$. Therefore

$$h(2m+1) = 2h(m) + 1 \, .$$

(For example, $h(23) = 15$.) We can now find a formula for $h(n)$. Let k be chosen such that $2^k \le n < 2^{k+1}$. Then

$$h(n) = 2(n - 2^k) + 1 \, . \qquad (*)$$

To see that this is true for all values of n, assume that it is true for some particular m. Then it is also true for $2m$ and $2m+1$, because

$$\begin{aligned} 2^k \le 2m < 2^{k+1} \quad &\Rightarrow \quad 2^{k-1} \le m < 2^k \\ &\Rightarrow \quad h(m) = 2(m - 2^{k-1}) + 1 \\ &\Rightarrow \quad h(2m) = 2h(m) - 1 = 2(2m - 2^k) + 1 \end{aligned}$$

and

$$\begin{aligned} 2^k \le 2m+1 < 2^{k+1} \quad &\Rightarrow \quad 2^{k-1} \le m < 2^k \\ &\Rightarrow \quad h(m) = 2(m - 2^{k-1}) + 1 \\ &\Rightarrow \quad h(2m+1) = 2h(m) + 1 = 2(2m + 1 - 2^k) + 1 \, . \end{aligned}$$

But $(*)$ certainly gives the correct answer for one student, so what we have just proved shows that it also gives the correct answer for 2 or 3 students, therefore also for $4, 5, 6$ or 7 students, and so on. Now we can easily answer the question.

(a) We have $1024 \le 1997 < 2048$ and so $h(1997) = 2(1997 - 1024) + 1 = 1947$. That is, the 1947th student goes home early.

(b) Let the number of students be n, where $2^k \le n < 2^{k+1}$. Then since $h(n) = 1997$ we have $2(n - 2^k) + 1 = 1997$ and so

$$n = 998 + 2^k \, .$$

However $n - 2^k < 2^{k+1} - 2^k = 2^k$, so $2^k > 998$ and $k \ge 10$. So the possible numbers of students in the school are

$$n = 998 + 2^{10} \, , \; 998 + 2^{11} \, , \; 998 + 2^{12} \, , \; \ldots = 2022 \, , \; 3046 \, , \; 5094 \, , \; \ldots \; .$$

6. How many lists of n numbers are there for which **both** of the following statements are true? –

- the list may contain only the integers $1, 2, 3, \ldots, k$; however, some numbers may be used more than once, and some not at all;

- no collection of two or more consecutive numbers fom the list adds up to a multiple of $k+1$.

Solution. Let $s_0 = 0$, let s_1 be the first number in the list, s_2 the sum of the first two, s_3 the sum of the first three, and so on: s_n is the sum of all the numbers in the list. Note that if we know the numbers in the list we can calculate the sums $s_0, s_1, s_2, \ldots, s_n$, and if we know these sums we can find the numbers in the list. In fact, since each number in the list can only be $1, 2, 3, \ldots$ or k, all we need to know is the remainder when the sums are divided by $k+1$. No two of these remainders may be the same, for then by subtraction we should obtain a collection of consecutive numbers from the original list whose sum has remainder zero when divided by $k+1$. Therefore in order to count the number of lists it is enough to count the number of possibilities for $s_1, s_2, \ldots, s_n$, where each is $1, 2, 3, \ldots$ or k and no two are the same. Clearly this is impossible if $n > k$; if $n \leq k$ then there are k possible choices for s_1, then $k-1$ for s_2, and so on, ending with $k+1-n$ choices for s_n. So the number of lists for which both the given statements are true is zero if $n > k$, and

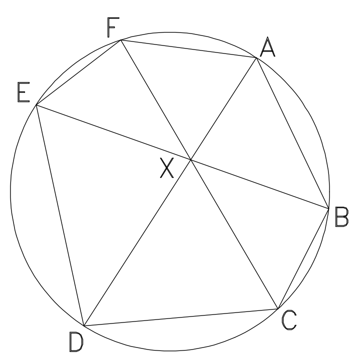
$$k(k-1)(k-2)\cdots(k+1-n)$$

if $n \leq k$.

SENIOR DIVISION

1. $ABCDEF$ is a hexagon inscribed in a circle, with the property that the lines AD, BE and CF meet at a single point. Prove that

$$AB \times CD \times EF = BC \times DE \times FA\,.$$



Solution. Let X be the point of intersection of the three lines. Triangles ABX and EDX are similar because $\angle AXB = \angle EXD$ (vertically opposite angles) and $\angle ABX = \angle EDX$ (angles standing on the same arc). Therefore

$$\frac{AB}{BX} = \frac{DE}{DX}\,;$$

similarly

$$\frac{CD}{DX} = \frac{FA}{FX} \quad \text{and} \quad \frac{EF}{FX} = \frac{BC}{BX} \, .$$

Multiplying these three equations,

$$\frac{AB}{BX} \times \frac{CD}{DX} \times \frac{EF}{FX} = \frac{DE}{DX} \times \frac{FA}{FX} \times \frac{BC}{BX}$$

and so

$$AB \times CD \times EF = BC \times DE \times FA \, .$$

2. (a) Solve the simultaneous equations

$$a + b + c = 2 \, , \quad bc + ac + ab = -5 \, , \quad abc = -6 \, .$$

(b) Find all real solutions of the simultaneous equations

$$a + b + c = -1 \, , \quad a^2 + b^2 + c^2 = 15 \, , \quad a^3 + b^3 + c^3 = -13 \, .$$

Solution.

(a) The numbers a, b and c are solutions of $(x-a)(x-b)(x-c) = 0$. By expanding the left hand side we have $x^3 - (a+b+c)x^2 + (bc+ac+ab)x - abc = 0$, and the given relations yield the equation

$$x^3 - 2x^2 - 5x + 6 = 0 \, .$$

This cubic is easily solved to give $x = 1, -2, 3$. Therefore a, b and c are $1, -2$ and 3, in any order.

(b) We have

$$(a + b + c)^2 = (a^2 + b^2 + c^2) + 2(bc + ac + ab) \, ,$$

that is, $1 = 15 + 2(bc + ac + ab)$, and so $bc + ac + ab = -7$. Also,

$$\begin{aligned} (a+b+c)(a^2+b^2+c^2) &= (a^3+b^3+c^3) + (a^2b + a^2c + b^2a + b^2c + c^2a + c^2b) \\ &= (a^3+b^3+c^3) + (a+b+c)(bc+ac+ab) - 3abc \, , \end{aligned}$$

so $-15 = -13 + 7 - 3abc$ and we find $abc = 3$. Now the method of part (a) leads to the cubic $x^3 + x^2 - 7x - 3 = 0$. The cubic has a solution $x = -3$, and therefore

$$(x + 3)(x^2 - 2x - 1) = 0 \, .$$

Solving the quadratic by completing the square (or by the quadratic formula) provides two more solutions $x = 1 \pm \sqrt{2}$. Therefore a, b and c are $-3, 1 + \sqrt{2}$ and $1 - \sqrt{2}$, in any order.

3. The students in a school are arranged in a circle and numbered $1, 2, 3, \ldots$. Starting with number 2, every second student is sent away to clean up part of the schoolyard, until only one student is left, and is allowed to go home early. (So, if there were eleven students, they would be chosen in the order $2, 4, 6, 8, 10, 1, 5, 9, 3, 11$ and the student to go home early would be number 7.)

   (a) If there are 1997 students in the school, which one gets to go home early?

   (b) How many students may there be in the school if number 1997 is to be the one who goes home early?

   Solution. See question 5 in the Junior Division.

4. In this problem $[x]$ denotes the greatest integer less than or equal to x; for example, $[\pi] = 3$. Let n be a positive integer. Find the smallest positive integer m such that
$$\left[\frac{n^2}{n+m}\right] = \left[\frac{n^2}{n+m+1}\right].$$

   Solution. The smallest m satisfying the given equation is the nearest integer to $\sqrt{n}$. We shall prove this in two steps, firstly showing that if m takes this value then the equation is true, and secondly that if m takes any smaller value then the equation is not true. Before we begin let's just note that the square root of an integer can never be an integer plus a half, so the difficulty of whether to round halves up or down to get the "nearest" integer does not arise in this problem.

   So, let m be the nearest integer to $\sqrt{n}$; then $\sqrt{n} - \frac{1}{2} < m < \sqrt{n} + \frac{1}{2}$. Now
$$\frac{n^2}{n+m} = \frac{(n^2 - m^2) + m^2}{n+m} = n - m + \frac{m^2}{n+m}. \qquad (*)$$

   Also,
$$m^2 - (n+m) = m(m-1) - n < \left(\sqrt{n} + \tfrac{1}{2}\right)\left(\sqrt{n} - \tfrac{1}{2}\right) - n = -\tfrac{1}{4} < 0\,,$$

   and hence $m^2 < n + m$. Therefore the right hand side of $(*)$ is an integer $n - m$ plus a fraction greater than 0 and less than 1; so
$$\left[\frac{n^2}{n+m}\right] = n - m\,.$$

   On the other hand,
$$\frac{n^2}{n+m+1} = \frac{(n-m)(n+m+1) + m^2 + m - n}{n+m+1} = n - m + \frac{m^2 + m - n}{n+m+1}. \qquad (**)$$

   Here we have
$$(m^2+m-n) - (n+m+1) = m^2 - 2n - 1 < \left(\sqrt{n} + \tfrac{1}{2}\right)^2 - 2n - 1 = -\tfrac{3}{4} + \sqrt{n} - n < 0\,.$$

Also

$$m^2 + m - n = m(m+1) - n > \left(\sqrt{n} - \tfrac{1}{2}\right)\left(\sqrt{n} + \tfrac{1}{2}\right) - n = -\tfrac{1}{4} \ ,$$

and since $m^2 + m - n$ is an integer it must be zero or positive. Putting these last two inequalities together we have

$$0 \le m^2 + m - n < n + m + 1 \ ,$$

and so $(**)$ again is an integer plus a fraction less than 1. Hence

$$\left\lfloor \frac{n^2}{n+m+1} \right\rfloor = n - m = \left\lfloor \frac{n^2}{n+m} \right\rfloor \ ,$$

and we have proved that the equation is true when m is the closest integer to $\sqrt{n}$. Now suppose that m has some smaller value; thus $m < \sqrt{n} - \frac{1}{2}$. As above we have

$$\left\lfloor \frac{n^2}{n+m} \right\rfloor = n - m \ .$$

If we reconsider $(**)$ we find that in this case

$$m^2 + m - n = m(m+1) - n < \left(\sqrt{n} - \tfrac{1}{2}\right)\left(\sqrt{n} + \tfrac{1}{2}\right) - n = -\tfrac{1}{4} < 0 \ .$$

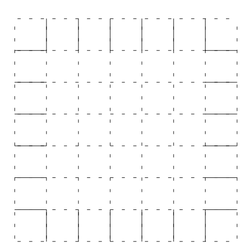
Therefore the right hand side of $(**)$ is now an integer *minus* something; by rounding each side downwards we have

$$\left\lfloor \frac{n^2}{n+m+1} \right\rfloor < n - m \ ,$$

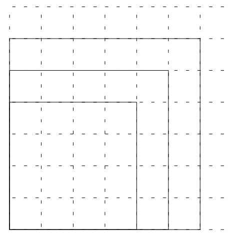
and the required equality is no longer true.

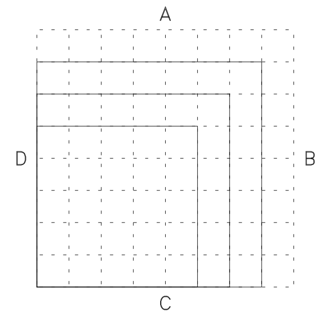
5. What is the least number of squares required in order that their edges cover **all** the edges of an $n \times n$ square grid?

   Solution. The accompanying diagram shows the grid for $n = 7$. Consider the edges indicated with solid lines. There are $4n - 4$ of them, and a little thought shows that any square can cover only two of the edges; therefore at least $2n - 2$ squares are required.



Now we must show that $2n-2$ squares are enough. First we assume that n is odd. The left-hand diagram below shows $\frac{1}{2}(n-1)$ squares, and these squares cover all horizontal edges in the top left quarter of the grid, and all vertical edges in the



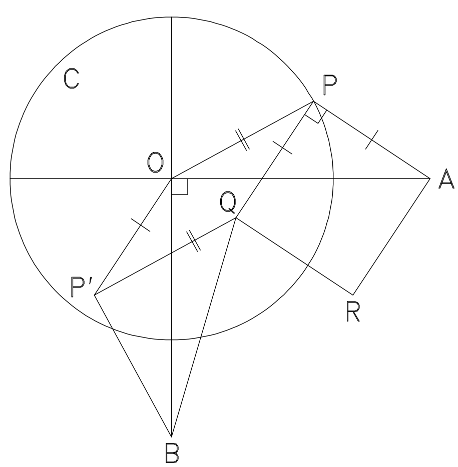


bottom right corner. By constructing similar patterns of squares in all four corners we obtain a total of $2n-2$ squares covering all the internal edges of the grid; and it is clear from the diagram that the outside border of the grid is also covered by these squares. If, on the other hand, n is even, a similar placement of $\frac{1}{2}(n-2)$ squares in each corner covers all edges of the grid except the central lines AC and BD. It is then easy to see from the diagram that the job is completed by two further squares: one with opposite vertices at A and B, the other with opposite vertices at C and D. The total number of squares used is thus $2(n-2)+2 = 2n-2$.

Note. These results are incorrect for $n = 1$ and $n = 2$. The question ought to have specified that $n \geq 3$. In fact it is easy to see that one square is required for $n = 1$ and three for $n = 2$.

6. Let A be a point outside a circle C. For any point P on C, let Q be the vertex opposite A on the square $APQR$. Determine the path traced out by Q as P moves around the circle C.

   Solution. Let O be the centre of C; construct B such that the lines OA and OB are perpendicular, and the distance OB is equal to the distance OA. Then Q traces out a circle with centre B and radius $\sqrt{2}$ times the radius of C.



   Proof. Given O, P and Q, construct P' such that $OPQP'$ is a parallelogram. Then we have $OA = OB$ (construction); also $AP = PQ$ (given) and $PQ = OP'$ (construction), so $AP = OP'$. Now since the angles of a triangle add up to $180°$, and so do two adjacent angles of a parallelogram, we have

$$\angle OAP + \angle APQ + \angle QPO + \angle POA = \angle BOP' + \angle BOA + \angle POA + \angle QPO\ .$$

Since $\angle APQ = \angle BOA = 90°$ this proves that $\angle OAP = \angle BOP'$, and so $\triangle OAP$ is congruent to $\triangle BOP'$ (two sides and included angle).

Hence $BP' = OP$. But by construction $OP = P'Q$, and so $BP' = P'Q$. Furthermore, $\angle BP'O = \angle OPA$, and since $\angle QP'O = \angle OPQ$ (opposite angles of a parallelogram are equal) we find that $\angle BP'Q = \angle QPA$. Thus $\triangle BP'Q$ is a right-angled isosceles triangle and we have

$$BQ = \sqrt{2}\, P'Q = \sqrt{2}\, OP$$

as claimed above. *Note*. If you labelled the square with vertices A, P, Q, R in clockwise order then you would have found B to be above the circle, instead of below as in the diagram. The rest of the solution would have been the same.