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SOME SIMPLE IDEAS ABOUT WAVES

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Introduction

A great many aspects of our lives involve waves of one sort or another. We speak to each other via sound waves, light and radio signals travel via electromagnetic waves. Waves at the beach are well-known to all of us and are just one of a great many types of waves in the atmosphere and ocean some of which have great influence on the weather, for example. Although these different types of waves are produced by different physical mechanisms, they all have many features in common and their properties can often be described using some quite simple mathematics.

Propagation in One Dimension

We will confine attention in this article to waves which are propagating in one dimension which we will call x. You can think of x as being distance measured along an optical fibre, or distance from a radio transmitter for example. If t denotes time, consider an expression of the form

$$\phi = f(x - ct) \tag{1}$$

where c is a positive constant, f is a function of a single variable (e.g. $f(x) = \sin 2x$ so $f(x - ct) = \sin 2(x - ct)$) and ϕ is the physical quantity of interest, for example the excess air pressure in a sound wave or the electric field in a radio wave. To understand the meaning of (1), consider an observer moving along the x axis with speed c towards $x = +\infty$. If this observer was at position x_1 at time t_0 then her position at time t will be $x(t) = x_1 + c(t - t_0)$ and so the value of ϕ which she sees will be

$$\phi_1 = f(x_1 + c(t - t_0) - ct) = f(x_1 - ct_0)$$

which depends only on the observer's position at time t_0. So, as far as this observer is concerned, ϕ never changes. Similarly, another observer who was located at x_2 at time t_0 would see $\phi_2 = f(x_2 - ct_0)$ which is dependent only upon this observer's initial position.

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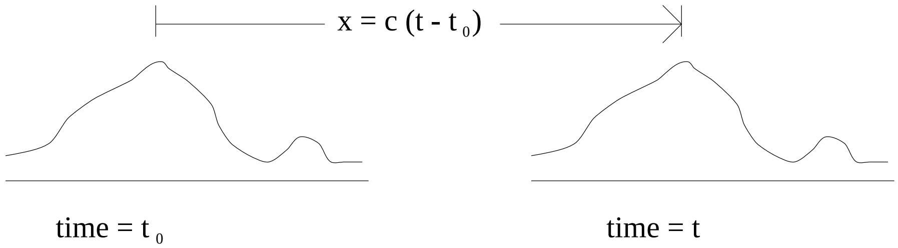

Figure 1: Illustrating how an expression of the form $\phi = f(x - ct)$ represents a disturbance progagating to the right without change of form when $c > 0$. In the time interval from t_0 to t the disturbance travels a distance $x = c(t - t_0)$.

Thus $\phi = f(x - ct)$ represents a disturbance propagating to the right with speed c without changing its shape. Similarly $\phi = f(x + ct)$ represents a disturbance propagating to the left without changing its shape. The first case is depicted in Figure 1.

Simple Harmonic Waves

A special case of the above type of disturbance concerns oscillatory periodic disturbances which may be described by expressions of the form

$$\phi(x, t) = A \cos k(x - ct + \theta)$$

where A, k, c and θ are constants. This can also be written $\phi(x, t) = A\cos(kx - \omega t + \epsilon)$ where $\omega = kc$ and $\epsilon = k\theta$. Now we know that the cosine function has period 2π so that $\cos(X \pm 2\pi) = \cos X$ for any X. (Remember that we use radians not degrees). Thus

$$\begin{aligned} \phi(x + 2\pi/k,\ t) &= A\cos(k(x + 2\pi/k) - \omega t + \epsilon) \\ = A\cos(kx + 2\pi - \omega t + \epsilon) &= A\cos(kx - \omega t + \epsilon) = \phi(x,\ t). \end{aligned}$$

Thus, ϕ has period $2\pi/k$ in x. Similarly $\phi(x, t + 2\pi/\omega) = \phi(x, t)$. It is convenient to define $\lambda = 2\pi/k$ and $T = 2\pi/\omega$. The names and units of the various quantities which have cropped up here are displayed in the following table.

Quantity	Name	Units
A	Amplitude	Depends on physical situation
k	Wavenumber	radians/metre
$\lambda = 2\pi/k$	wavelength	metres
ω	frequency	radians/second
$T = 2\pi/\omega$	period	seconds
ϵ	phase angle	radians
$c = \omega/k$	wave speed	metres/second

You may be more familiar with the definition of frequency in terms of Hertz, i.e. cycles per second. If f is the frequency in Hertz, then $\omega = 2\pi f$. Thus if $f = 50$ Hertz, $\omega = 100\pi$ radians/sec. The units of A will depend on the situation we are studying. For water waves, ϕ might be the height of the instantaneous water surface above still-water level, so A would be measured in metres. For sound waves, ϕ is a pressure so is measured in Newtons/metre^2. The meanings of λ and T are shown in Figures 2 and 3 respectively. The first can be thought of as a snapshot at fixed t . The variable ϕ is then periodic in x with period λ . The second represents measurements made at a fixed position x. The variable ϕ is then periodic in time with period T.

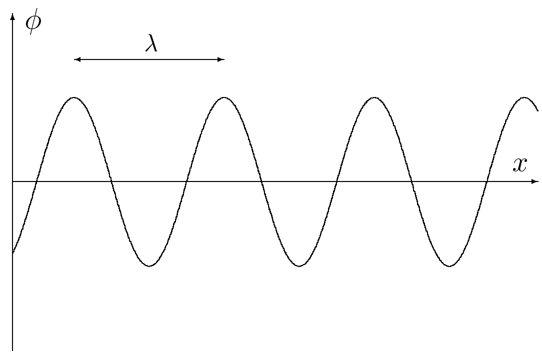

Figure 2: Illustrating the meaning of wavelength λ .

The quantity c is called the wave speed or wave velocity or phase velocity or phase speed or "celerity" (the terms "speed" and "velocity" are commonly used interchangably in this context, although that usage is not strictly correct). You can think of c as the speed with which an individual wave crest propagates along. (Imagine standing on a cliff and watching ocean waves propagating towards the shore. Keep your eyes on a particular crest as it moves. The speed with which it moves will be c).

Wave Dispersion

There are two fundamentally different types of wave motions:

i) Non-dispersive in which c is a constant

ii) Dispersive in which c depends on the wavelength, or equivalently on the frequency, of the waves.

Now the relation between the wavelength and the wave frequency depends on the physical situation we are talking about. In general, we will have $\omega = W(k)$ where the function W depends on the physics. This is called the **dispersion relation**. If $W(k) = Ck$ where C is constant, then the wave speed is

$$c = W/k = Ck/k = C.$$

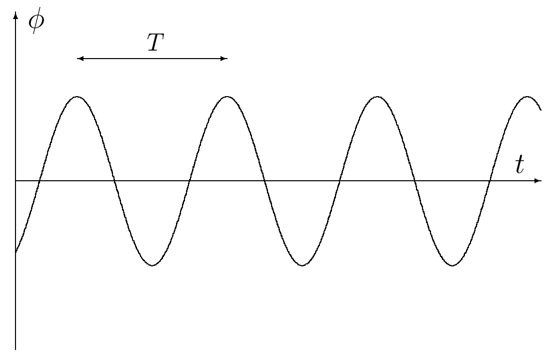

Figure 3: Illustrating the meaning of wave period T.

In this case, all waves travel with the same speed $c = C$ independent of their wavelength. Some examples of these are sound waves in a room for which $c \simeq 330$ metres/second (of course this depends on the atmospheric temperature and pressure) and electromagnetic waves in air for which $c \simeq 3 \times 10^8$ metres/second. For us it is important that sound and electromagnetic waves are non-dispersive. When we speak, we produce sounds of many different frequencies at the same time. If these all travelled at different speeds, our words would become less and less intelligible the further they travelled. Similar comments apply to radio waves.

Rather more interesting are dispersive waves in which c depends on k. In many, but not all, types of wave motion, c is a decreasing function of k so that long waves (small k, so large $\lambda = 2\pi/k$) travel faster than short waves (large k, so small λ).

One common example of this type of wave motion is water waves for which

$$c^2 = \frac{g}{k}\,\frac{e^{kH} - e^{-kH}}{e^{kH} + e^{-kH}}\ .$$

Here, H is the (constant) water depth and g is the acceleration due to gravity. When kH is large (which means wavelength <<depth) it can be shown that

$$c^2 \simeq \frac{g}{k} = \frac{g\lambda}{2\pi}.$$

[Try and prove this yourself by recalling the behaviour of e^x and e^{-x} as $x \to \infty$]. So for "short" waves the wave speed c is proportional to the square root of the wavelength. When kH is small (wavelength >> depth) it can be shown that

$$c^2 \simeq gH$$

which means that all long waves travel at the same speed $\sqrt{gH}$. A graph of $c/\sqrt{gH}$ as a function of kH is given in Figure 4. This is an example of what is termed a non-dimensional plot. Both c and $\sqrt{gH}$ are speeds, so $c/\sqrt{gH}$ has no dimensions and is independent of the system of units employed. Similarly for kH.

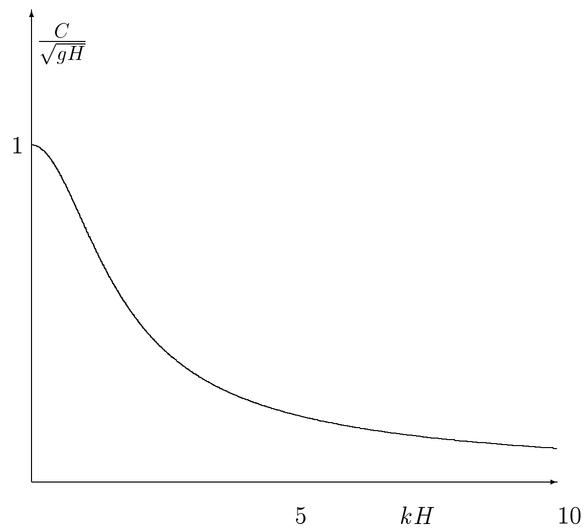

Figure 4: The non-dimensional phase speed $c/\sqrt{gH}$ of water waves as a function of $kH = 2\pi H/\lambda$ where H is the water depth and λ the wavelength.

Group Velocity

We have seen how the individual wave crests propagate at speed c, the phase velocity. In any real situation, we never have just a single frequency or wavenumber present, but a spread of them. This introduces a second velocity – the group velocity. To understand how this arises, consider two waves of the same amplitude

$$\phi = A\cos(k_1 x - \omega_1 t + \epsilon_1) + A\cos(k_2 x - \omega_2 t + \epsilon_2)\ .$$

Using $\cos(a-b) + \cos(a-b) = 2\cos a\cos b$ we can write this as

$$\begin{aligned} \phi &= 2\cos\left(\frac{k_1+k_2}{2}x - \frac{\omega_1+\omega_2}{2}t + \frac{\epsilon_1+\epsilon_2}{2}\right) \\ &\times \cos\left(\frac{k_1-k_2}{2} - \frac{\omega_1-\omega_2}{2}t + \frac{\epsilon_1-\epsilon_2}{2}\right). \end{aligned}$$

This consists of the **product** of two waves one travelling at speed $c_W = \dfrac{\omega_1+\omega_2}{k_1+k_2}$ and the other at $c_G = \dfrac{\omega_1-\omega_2}{k_1-k_2}$.

Suppose now that ω_1 and ω_2 are close together, and since ω_1 and ω_2 both satisfy the dispersion relation, k_1 and k_2 are also close together. So put $\omega_1 = \omega_2 + \delta\omega,\ k_1 = k_2 + \delta k$ where $\delta\omega$ and δk are both small.

Then

$$c_W = \frac{\omega_2+\omega_2+\delta\omega}{k_2+k_2+\delta k}\ =\ \frac{\omega_1+\omega_1-\delta\omega}{k_1+k_1-\delta k} \simeq \frac{\omega_1}{k_1} \simeq \frac{\omega_2}{k_2} = c$$

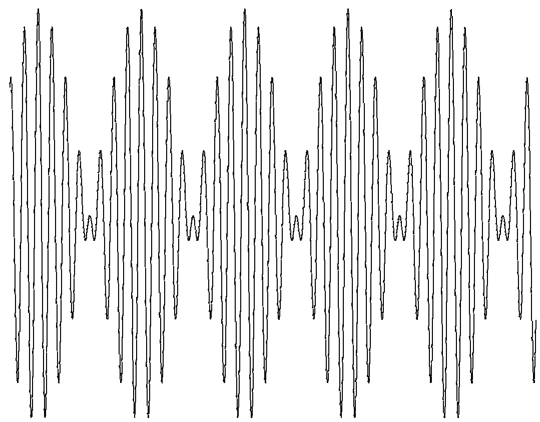

Figure 5: A plot of $\phi = A\cos(k_1 x - \omega_1 t + \epsilon_1) + A\cos(k_2 x - \omega_2 t + \epsilon_2)$ when $k_1 \simeq k_2$ and $\omega_1 \simeq \omega_2$.

and

$$c_G = \frac{\omega_2 + \delta\omega - \omega_2}{k_2 + \delta k - k_2} = \frac{\delta\omega}{\delta k} \simeq \frac{d\omega}{dk}$$

The physical picture is thus one of a "carrier wave"

$$\cos\left(\frac{k_1 + k_2}{2}x - \frac{\omega_1 + \omega_2}{2}t + \frac{\epsilon_1 + \epsilon_2}{2}\right)$$

(which propagates at speed $c_W \simeq \frac{\omega_1}{k_1} \simeq \frac{\omega_2}{k_2}$) and has a varying amplitude given by

$$A(x,t) = 2\ \cos\left(x\frac{\delta k}{2} - t\frac{\delta\omega}{2} + \frac{\epsilon_1 - \epsilon_2}{2}\right).$$

This amplitude has wavelength $2\pi/(\delta k/2) = \frac{4\pi}{\delta k}$ which is very long compared with the wavelength $\frac{4\pi}{k_1 + k_2}$ of the carrier wave and travels at speed $c_G = \frac{\delta\omega}{\delta k}$. When δk is small, $c_G \simeq \frac{d\omega}{dk} = c_g$. The situation is shown in Figure 5 .

These ideas are readily extended to situations where we have waves of many different frequencies present, not necessarily all of the same amplitude. If all the waves have frequencies close to some central frequency ω_0 and $\omega_0 = W(k_0)$ is the dispersion relation, then the picture which emerges is

$$\phi = A(x,t)\cos(k_0 x - \omega_0 t + \epsilon_0) \tag{2}$$

where $A(x,t)$ is the slowly-varying amplitude which propagates at speed

$$c_g = \frac{d\omega}{dk}$$

evaluated at $k = k_0$. This speed is called the **group speed** or group velocity, and in many ways is more fundamental than the phase velocity since it is generally the speed with which the **energy** of the wave propagates. If the waves are non-dispersive, then the dispersion relation is $\omega = ck$ where c is a constant. Then $c_g = \frac{d\omega}{dk} = c$ so the phase and group speeds are the same. If the waves are dispersive, then c depends on k so

$$c_g = \frac{d}{dk}(kc) = c + k\frac{dc}{dk}.$$

This is **different** from c. This has the surprising consequence that although the individual wave crests are travelling at speed c, the energy of the disturbance is travelling at a different speed c_g. The expression (2) is sometimes referred to as a *amplitude-modulated wavetrain* and is depicted in Figure 6. The individual wave crests travel at the phase speed $c = \omega_0/k_0$ but the amplitude pattern travels at the group speed c_g.

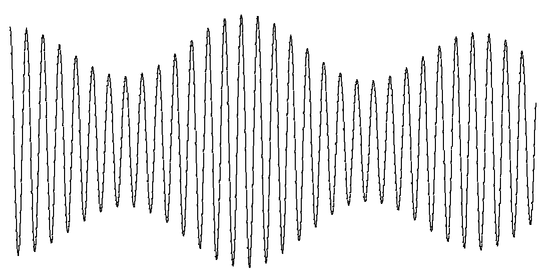

Figure 6: An amplitude-modulated wavetrain given by (2)

Some Examples

1. Deep water waves for which $\omega^2 = gk$. Then $\omega = \sqrt{gk}$ so $\frac{d\omega}{dk} = \frac{1}{2}\sqrt{g/k} = \frac{1}{2}\frac{\omega}{k} = \frac{1}{2}c$. Thus the group velocity of deep water waves is only half their phase velocity. Observations of storm-generated waves at sea confirm this – the time taken for waves of period T to travel a distance L from their region of generation to a distant observing station is not L/c but rather L/c_g where c_g is the group velocity of waves of period T. Thus, it takes the waves twice as long to reach the observing station than one would naively expect.

2. Rossby Waves. These are special kind of long period wave in the atmosphere and ocean and are due to the rotation of the earth. If x is measured eastwards, the dispersion relation is

$$\omega = \frac{-\beta k}{k^2 + \delta^2}$$

where β and δ are positive physical constants. For these waves

$$c = \frac{\omega}{k} = \frac{-\beta}{k^2 + \delta^2}.$$

Thus, $c < 0$ so the wave crests travel always towards the west.

However $c_g = \dfrac{d\omega}{dk} = \dfrac{-\beta(k^2 + \delta^2 - 2k^2)}{(k^2 + \delta^2)^2} = \beta \dfrac{(k^2 - \delta^2)}{(k^2 + \delta^2)^2}$.

Thus, $\dfrac{d\omega}{dk}$ is positive if $k^2 > \delta^2$, i.e. $\dfrac{2\pi}{k} < \dfrac{2\pi}{\delta}$ and is negative if $k^2 < \delta^2$, i.e. if $\dfrac{2\pi}{k} > \dfrac{2\pi}{\delta}$. Thus, long waves (those with wavelength $\lambda > 2\pi/\delta$) have group velocities in the same direction as the phase velocity but short waves (those with wavelengths $\lambda < 2\pi/\delta$) have group velocities opposite to the phase velocities. For these short waves, the individual wave crests are propagating towards the west, yet the energy is propagating towards the east. This seemingly bizarre behaviour is not atypical of many types of waves which occur in the atmospere and ocean. For example, there are other types of waves called **internal waves** which are due to density variations in the atmosphere or ocean. For these internal waves, the phase and group velocities are at right angles to each other.

Summary

Simple waves may be described by well-known trigonometric functions. Two fundamentally different velocities emerge – the phase velocity which tells us how individual wave crests travel and the group velocity which tells us how the energy of the wave disturbance as a whole travels. These two velocities are in general different and, indeed, may nor even be in the same direction. These ideas underlie many applications in meteorology, civil engineering, communications technology and many other fields in which waves play a role.

Exercise

Very short water waves are influenced by surface tension – the force that enables mosquitoes to stand and walk on the surface of water. If the water is deep, the dispersion relation can be shown to be

$$\omega^2 = gk + \mathcal{F}k^3$$

where $\mathcal{F}$ is a physical constant which measures the strength of the surface tension.

a) Show that the phase speed has a minimum when $k = \sqrt{g/\mathcal{F}} = k_m$.

b) Evaluate the wavelength $\lambda_m = 2\pi/k_m$ and the minimum phase speed when $g = 9.8\text{m/sec}$ and $\mathcal{F} = 0.74 \times 10^{-4}\text{metre}^3/\text{sec}^2$.

(Ans: $\lambda_m \simeq 1.726 \times 10^{-2}\text{m.}$, i.e. 1.726 cm and $c_{\min} = 2.32 \times 10^{-1}$ metre/sec).

c) Find the group speed and show that it is less than the phase speed when $\lambda > \lambda_m$ but greater than the phase speed when $\lambda < \lambda_m$. Very short waves (i.e. those with $\lambda < \lambda_m$) are called *capillary waves*. For them, the restoring force of surface tension is more important than that of gravity. Try to show that $c_g \to 3c/2$ as $\lambda \to 0$.