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SOLUTIONS TO PROBLEMS 1001-1006

Q.1001 On the island described in question 2 of the Junior Division for this year's mathematics competition, each town entered one Australian Rules football team in the national championship. The teams are nicknamed the Bears, Cats, Swans, Magpies, Tigers and Hawks – just a meaningless coincidence, of course. One game was played between any two teams, with the winner scoring four points and the loser none, or two each in case of a draw. The eventual winners were the Magpies, with a score of 18 points. The only team they did not beat was the Hawks, who scored just 4 points; in fact the Hawks were the only team not to win a game. The runners-up were the Cats with 12 points, followed by the Swans and Tigers with 10 points each; these three teams lost just one game each. Given that the Tigers did not beat the Bears, find the results of all fifteen games.

By another meaningless coincidence, two teams drew their match if and only if their home towns are connected directly by a road. Which team came from which town?

ANS. First we determine how many games were won, drawn and lost by each team. It is clear that the Magpies had 4 wins, 1 draw and no losses; since the Hawks scored 4 points without winning a game they must have drawn 2 matches and lost 3. Since the Cats lost just one match in scoring 12 points they must have had 2 wins and 2 draws, and similarly the Swans and the Tigers each scored 1 win, 3 draws and 1 loss. If we add up what we have found so far we find 8 wins, 11 draws and 6 losses altogether. Now the Bears won (at least) one game since we are told that "the Hawks were the only team not to win a game". So the Bears must have lost 3 games, as the total number of games won by all sides must obviously equal the total lost. To complete their 5 games, the Bears must have played 1 draw. So the final competition ladder would have looked as follows.

	wins	draws	losses	points
Magpies	4	1	0	18
Cats	2	2	1	12
Swans	1	3	1	10
Tigers	1	3	1	10
Bears	1	1	3	6
Hawks	0	2	3	4

As for the results of the individual matches, the Magpies drew with the Hawks and beat the other four teams. The Tigers didn't beat the Bears, and didn't lose to them either (they lost only one game, to the Magpies), so the Tigers and the Bears drew their match. There are now only three teams which could have drawn with the Swans, and the other results are easily found. The complete table of results is

- the Magpies beat the Cats, the Swans, the Tigers and the Bears and drew with the Hawks;
- the Cats beat the Bears and the Hawks, drew with the Swans and the Tigers and lost to the Magpies;
- the Swans beat the Bears, drew with the Cats, the Tigers and the Hawks and lost to the Magpies;
- the Tigers beat the Hawks, drew with the Cats, the Swans and the Bears and lost to the Magpies;
- the Bears beat the Hawks, drew with the Tigers and lost to the Magpies, the Cats and the Swans;
- the Hawks drew with the Magpies and the Swans and lost to the Cats, the Tigers and the Bears.

The teams belonging to the various towns are

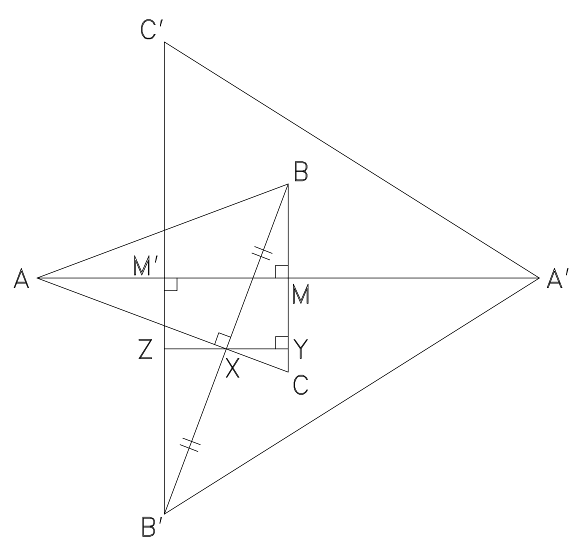
A: Bears,	B: Hawks,	C: Magpies,
D: Cats,	E: Tigers,	F: Swans.

Q.1002 Extend problem 4 in the Junior Division of the competition to the case where $\triangle ABC$ is isosceles. Find the ratio

$$\text{area}\,(\triangle A'B'C')/\text{area}\,(\triangle ABC)$$

in terms of the base angles of $\triangle ABC$.

ANS.



Triangles $\triangle ABC$ and $\triangle A'B'C'$ are shown in the above diagram. For convenience write $h = AM$, $b = BM = CM$ and $\alpha = \angle ACB$. Note that

$$h = b \tan \alpha$$

and that

$$\angle XBC = 90 - \alpha$$

with angles measured in degrees. Since $\triangle BXY$ and $\triangle BXC$ are right-angled we have

$$\begin{aligned} BX &= BC \sin \alpha = 2b \sin \alpha \\ XY &= BX \sin \angle XBY = 2b \sin \alpha \cos \alpha \\ BY &= BX \cos \angle XBY = 2b \sin^2 \alpha \ . \end{aligned}$$

It's easy to see that $\triangle BXY$ and $\triangle B'XZ$ are congruent. Therefore the altitude of $\triangle A'B'C'$ is

$$\begin{aligned} M'M + MA' &= ZX + XY + MA' = 2\,XY + h \\ &= (4 \sin \alpha \cos \alpha + \tan \alpha) \ . \end{aligned}$$

Similarly the base of $\triangle A'B'C'$ is

$$\begin{aligned} 2\,B'M' &= (BY + ZB' - BM) = 2(2\,BY - BM) \\ &= b(4 \sin^2 \alpha - 1) \ . \end{aligned}$$

Therefore the ratio of the areas is

$$\begin{aligned} \frac{\text{area}(\triangle A'B'C')}{\text{area}(\triangle ABC)} &= \frac{b^2(4 \sin \alpha \cos \alpha + \tan \alpha)(4 \sin^2 \alpha - 1)}{b^2 \tan \alpha} \\ &= (4 \cos^2 \alpha + 1)(4 \sin^2 \alpha - 1). \end{aligned}$$

Q.1003

(a) In how many ways can we choose n integers $x_1, x_2, \ldots, x_n$ such that each is $0, 1$ or 2 and their sum is even?

(b) What if the numbers are to be $1, 2$ or 3 and their sum even?

ANS.

(a)Let $f(n)$ be the number of ways of choosing n integers, each $0, 1$ or 2, so that their sum is even. Then the number of ways of choosing $n + 1$ integers under the same conditions is $f(n + 1)$. But we can determine the number of ways of choosing the $n + 1$ integers in another way. First choose x_{n+1}. Then

- if x_{n+1} is even (two possibilities) choose the other n integers such that their sum is even ($f(n)$ possibilities);

- if x_{n+1} is odd (one possibility) choose the other n integers such that their sum is odd ($3^n - f(n)$ possibilities).

Therefore

$$f(n+1) = 2f(n) + 3^n - f(n) = f(n) + 3^n \ , \qquad (*)$$

and it is clear that $f(1) = 2$. Since $(*)$ is true for all $n \geq 1$ we have

$$\begin{aligned} f(n) &= f(n-1) + 3^{n-1} \\ &= f(n-2) + 3^{n-2} + 3^{n-1} \\ &= f(n-3) + 3^{n-3} + 3^{n-2} + 3^{n-1} \\ &= \cdots \\ &= f(1) + 3^1 + \cdots + 3^{n-3} + 3^{n-2} + 3^{n-1} \\ &= 2 + 3^1 + \cdots + 3^{n-3} + 3^{n-2} + 3^{n-1} \ ; \end{aligned}$$

by summing the geometric series and simplifying we find that the number of ways of choosing the n integers is

$$f(n) = \frac{3^n + 1}{2} \ .$$

(b)Let $g(n)$ be the number of ways of choosing n integers, each $1, 2$ or 3, so that their sum is even. As in (a) we find

$$g(n+1) = g(n) + 2(3^n - g(n)) = 2 \times 3^n - g(n)$$

with $g(1) = 1$, and so

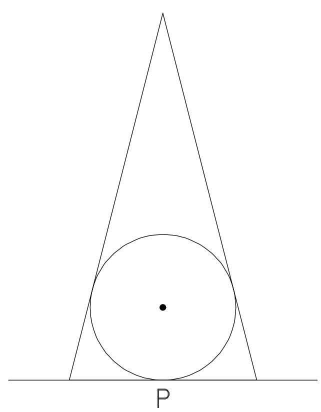
$$\begin{aligned} g(n) &= 2 \times 3^{n-1} - g(n-1) \\ &= 2 \times 3^{n-1} - 2 \times 3^{n-2} + g(n-2) \\ &= \cdots \\ &= 2 \times 3^{n-1} - 2 \times 3^{n-2} + \cdots \pm 2 \times 3^1 \mp g(1) \ , \end{aligned}$$

where the $\pm$ sign is $+$ if n is even and $-$ if n is odd. We can therefore write

$$\begin{aligned} g(n) &= 2 \times 3^{n-1} - 2 \times 3^{n-2} + \cdots \\ &\qquad + 2 \times (-1)^n \times 3 - (-1)^n \\ &= 2 \times (-1)^n \times 3 \, \frac{(-3)^{n-1} - 1}{(-3) - 1} - (-1)^n \\ &= \frac{3^n + 3 \times (-1)^n}{2} - (-1)^n \\ &= \frac{3^n + (-1)^n}{2} \ . \end{aligned}$$

Q.1004 Is it possible to circumscribe two non-congruent triangles of equal area around the same circle?

ANS. Yes it is. One way to see this is to consider the diagram below. Keep the circle fixed, and consider isosceles triangles circumscribed around the circle, with bases tangent to the circle at P. Imagine that the height of the triangle is very large; then the area of the triangle must also be very large as its base can never be less than the diameter of the circle. If the height begins to decrease the area will also begin to decrease; however, at a certain point the area must reach a minimum value and then begin to increase, because if the height is very small (just slightly bigger than the diameter of the circle) then the base will be large and the area will again be very large. Since, as we vary the height, the area of the triangle decreases to a minimum and then increases again, there must be many pairs of triangles with different heights and the same areas; but these triangles are not congruent since the base angles are not the same.



(If you have studied calculus you may like to find the height which gives a triangle of minimum area.)

Q.1005 Prove that if p and q are odd numbers then $p^2 - 4q$ is not a square.

ANS. Suppose that $p^2 - 4q = r^2$. If p and q are odd then r is also odd and we may write $p = 2s - 1$, $q = 2t - 1$ and $r = 2u - 1$. Then we have

$$(2s-1)^2 - 4(2t-1) = (2u-1)^2 ,$$

that is,

$$4s(s-1) - 8t + 5 = 4u(u-1) + 1 .$$

Since either s or $s - 1$ must be even, the first term on the left hand side

is a multiple of 8; similarly, the first term on the right hand side is a multiple of 8. Therefore, dividing each side by 8 leaves a remainder of 5 on the left hand side and 1 on the right hand side. Therefore the two sides cannot in fact be equal, and $p^2 - 4q$ cannot be a square.

Q.1006 Show that 343 is a factor of $2^{147} - 1$.

ANS. Note that $343 = 7^3$. We have

$$\begin{aligned} 2^{147} - 1 &= 8^{49} - 1 \\ &= (7+1)^{49} - 1 \\ &= 7^{49} + \binom{49}{1} 7^{48} + \cdots + \binom{49}{2} 7^2 + \binom{49}{1} 7^1 + 1 - 1\,; \end{aligned}$$

the last two terms cancel and every other term is divisible by 7^3, and this shows that $2^{147} - 1$ is a multiple of 343.