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HOW TO CONSTRUCT REGULAR 7-SIDED POLYGONS — AND MUCH ELSE BESIDES

Part 1 — The Basic Construction

by

Peter Hilton[1] **and Jean Pedersen**[2]

1. Introduction

Those of you who have taken plane geometry will know that the Greeks were fascinated with the challenge of constructing regular polygons — that is, those polygons with all sides of the same length and all angles equal. We will refer to such regular N-sided polygons simply as regular N-gons. The Greeks wanted to create these polygons by using what is called a Euclidean construction, that is, by using only an unmarked straight edge and a compass. The Greeks (ca. 350 B. C.) were successful in devising a Euclidean construction for an N-gon where

$$N = 2^c N_0, \text{ with } N_0 = 1,\ 3,\ 5, \text{ or } 15 \text{ and } c \geq 0.$$

Of course, we need $N \geq 3$ for the polygon to exist at all!

This is as far as the Greeks were able to go with their constructions and, in fact, it appears that no one else was able to make much more progress until about 2000 years later when Gauss (1777-1855) completely settled the questions inherent in the original problem. Gauss proved that a Euclidean construction of a regular N-gon is possible if and only if the number of sides N is of the form $N = 2^c p_1 p_2 p_3 \cdots p_k$, where $c \geq 0$ and the p_i are distinct Fermat primes — these are primes of the form $F_n = 2^{2^n} + 1$.

Gauss's discovery was remarkable — and, of course, it tells us precisely which N-gons admit a Euclidean construction, provided we know which Fermat numbers F_n are prime. Now it turns out that not all Fermat numbers are prime. Euler (1707 - 1783) showed that $F_5 = 2^{2^5} + 1$ is not prime, and, in fact, to this day, although many composite Fermat numbers have been identified, the only known prime Fermat numbers are

$$F_0 = 3,\ \ F_1 = 5,\ \ F_2 = 17,\ \ F_3 = 257,\ \ F_4 = 65537.$$

Thus we see that, even with Gauss's contribution, a Euclidean construction of a regular N-gon is known to exist for only a finite number of values of N, and even for these N we do not in all cases know an explicit construction.

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[1]Department of Mathematical Sciences, State University of New York, Binghamton, Binghamton, New York 13902-6000, U.S.A. and
Department of Mathematics, University of Central Florida, Orlando, Florida 32816-1364, U. S. A.
[2]Department of Mathematics, Santa Clara University, Santa Clara, California 95053, U. S. A.

Despite our knowledge of Gauss's work, and despite the availability of computers that can easily produce these regular polygons in a state referred to as virtual reality, we would like to know how to construct for ourselves, with our own hands, using easily available materials from the real world, all regular N-gons.

In this article (and its sequel) we will show you that, slightly redefining the problem formulated by the Greeks, you will be able, in principle, to construct, for any given value of N, a polygon that will be an arbitrarily good approximation to a regular N-gon. Furthermore, all this can done by a systematic and explicit paper-folding procedure that we will describe in detail, which depends, as you might expect, only on the precise value of N.

We should say something here about our methods of construction. We argue that in practice the approximations we obtain by folding paper are quite as accurate as the real world constructions obtained with a straight edge and compass — for the latter are only perfect in the mind. In both cases the real world result is a function of human skill, but our procedure, unlike the Euclidean procedure, is very forgiving, in that it tends to reduce the effects of human error — and, for most people, it is far easier to bisect an angle by folding paper than by Even when geometric figures are obtained from the best of modern-day computers their accuracy depends on the precision of the computer calculation and the resolution of the printer.

We will now get started on the paper-folding. It turns out that this will naturally lead us into some interesting number theory as well, but we will have to postpone that to the second part of our article in the next issue of Parabola.

2. The FAT Algorithm

We now explain a precise and fundamental folding procedure, involving a straight strip of paper with parallel edges (adding machine tape or ordinary unreinforced gummed tape work well). This procedure makes the top edge of the strip describe the sides of a regular N-gon, where $N \geq 3$.

For the moment assume that we have a straight strip of paper that has creases or folds along straight lines emanating from vertices at the top edge of the strip. Further assume that the creases at those vertices, labelled A_n $(n = 0, 1, \cdots)$, form identical angles of[3] $\dfrac{\pi}{N}$ with the top edge (as shown in Figure 1(a)). Suppose further that these vertices are equally spaced. If we fold this strip on the line marked A_nC_n, as shown in Figure 1(b) (with $n = 0$), and then, with a twisting motion, fold the tape again, this time on the line marked A_nB_n, as shown in Figure 1(b), the direction of the top edge will be rotated through an angle of $\frac{2\pi}{N}$. We call this process, of **F**olding **A**nd **T**wisting, the FAT-algorithm.

Now observe that if the FAT-algorithm is performed at a sequence of N vertices A_n for $n = 0, 1, 2, \cdots, N-1$, then the top edge of the tape will have turned through

[3]If you are not familiar with radian measure all you need to know here is that $\pi = 180^\circ$ —and in the context of this article it may be useful to simply think of π as representing the straight angle. Thus $\frac{\pi}{2}$ represents a right angle, $\frac{\pi}{3}$ represents 60°, and 2π stands for 'all the way round the circle'.

an angle of 2π, so that the point A_N will come into coincidence with A_0. Thus the top edge will form a regular N-gon! Figure 2 illustrates a portion of the regular N-gon formed using the FAT-algorithm.

You might like to practise the FAT-algorithm by constructing a regular convex 8-gon.[4] Figure 3(a) shows a straight strip of paper on which the dotted lines indicate certain, theoretically exact, crease lines. In fact, these crease lines occur at equally spaced intervals along the top of the tape so that the angles occurring at the top of each vertical line are (from left to right) $\frac{\pi}{2}$, $\frac{\pi}{4}$, $\frac{\pi}{8}$, $\frac{\pi}{8}$. Figuring out how to fold a strip of tape to obtain this arrangement of crease lines should be an interesting exercise for any student who has had a course in plane geometry (complete step-by-step instructions are given in [1]). What interests us here is the observation that this tape has, at equally spaced intervals along the top edge, adjacent angles each measuring $\frac{\pi}{8}$, and we can therefore execute the FAT-algorithm at 8 consecutive vertices along the top of the tape to produce the regular octagon shown in Figure 3(b). (Of course, in constructing the model one would cut the tape on the first vertical line and glue a section at the end to the beginning so that the model would form a closed polygon.)

Notice that the tape shown in Figure 3(a) also has suitable crease lines that make it possible to use the FAT-algorithm to fold a square. We leave this as an exercise for those who are interested and turn to a more challenging construction.

Now, since the regular convex 7-gon is the first polygon we encounter for which we do not have available a Euclidean construction, we are faced with a real difficulty in making available a crease line making an angle of $\frac{\pi}{7}$ with the top edge of the tape. We proceed by adopting a general policy, that we will eventually say more about — we call it our optimistic strategy. Figure 4 shows the step-by-step folding of the appropriate tape. The rationale is as follows. Assume that we can crease an angle of $\frac{2\pi}{7}$ (certainly we can come close) as shown in Figure 4 (part 3). Then it is a trivial matter to fold the top edge of the strip DOWN to bisect this angle, producing two adjacent angles of $\frac{\pi}{7}$ at the top edge as shown in Figure 4(part 5). (We say that $\frac{\pi}{7}$ is the putative angle on this tape.) Then, since we are content with this arrangement, we go to the bottom of the tape where we observe that the angle to the right of the last crease line is $\frac{6\pi}{7}$ — and we decide, as paper folders, that we will always avoid leaving even multiples of π in the numerator of any angle next to the edge of the tape, so we bisect this angle of $\frac{6\pi}{7}$, by bringing the bottom edge of the tape UP to coincide with the last crease line as shown in Figure 4(part 6). We settle for this (because we are content with an odd multiple of π in the numerator) and go to the top of the tape where we observe that the angle to the right of the last crease line is $\frac{4\pi}{7}$ — and, since we have decided against leaving an even multiple of π in any angle next to an edge of the tape, we are forced to bisect this angle twice, by folding DOWN, as shown in Figure 4(parts 8,9,10),obtaining the arrangement of crease lines shown in Figure 4(part 11).

Now we notice that something miraculous has occurred! If we had really started with

[4] The point here is to practice folding paper and executing the FAT-algorithm, since we know that you are probably already familiar with a Euclidean construction of the regular octagon.

an angle of $\frac{2\pi}{7}$, and if we now continue introducing crease lines by repeatedly folding the tape DOWN twice at the top and UP once at the bottom, we get precisely what we want, namely, pairs of adjacent angles, measuring $\frac{\pi}{7}$, at points at equally spaced intervals along the top edge of the tape.[5] Let us call this folding procedure the $\mathbf{D^2U^1}$-folding procedure (or, more simply the $(\mathbf{2},\mathbf{1})$-folding procedure) and call the strip of creased paper it produces $\mathbf{D^2U^1}$-tape (or, again more simply, $(\mathbf{2},\mathbf{1})$-tape).

We suggest that before reading further you get a piece of paper and fold an acute angle which you call an approximation to $\frac{2\pi}{7}$. Then fold about 40 triangles using the D^2U^1-folding procedure, throw away the first 10 triangles, and try to construct the FAT 7-gon shown in Figure 5(b). You may then believe that the D^2U^1-folding procedure produces

tape on which the angles approach the values indicated in Figure 5(a). But, how do we prove that this evident convergence takes place? First, let's admit that the first angle folded down from the top of the tape as shown in Figure 4(part 3) might not have been precisely $\frac{2\pi}{7}$. Then the bisection forming the next crease would make the two acute angles nearest the top edge in Figure 4(part 5) only approximately $\frac{\pi}{7}$; let us call them $\frac{\pi}{7}+\epsilon$ (where ϵ may be either positive or negative)[6]. Consequently the angle to the right of this crease, at the bottom of the tape, would measure $\frac{6\pi}{7}-\epsilon$. When this angle is bisected, by folding up, the resulting two new acute angles nearest the bottom of the tape in Figure 4(part 7) would each measure $\frac{3\pi}{7}-\frac{\epsilon}{2}$, forcing the angle to the right of this crease line at the top of the tape to have measure $\frac{4\pi}{7}+\frac{\epsilon}{2}$. When this last angle is bisected twice by folding the tape down the two acute angles nearest the top edge of the tape, as shown in Figure 4(part 11), will measure $\frac{\pi}{7}+\frac{\epsilon}{8}$. This should make it clear that every time we repeat a D^2U^1-folding on the tape the error is reduced by a factor of 8.

Now it should be clear how our optimistic strategy has paid off. By assuming we have an angle that, when bisected, gives an angle of $\frac{\pi}{7}$ at the top of the tape to begin with, and folding accordingly, we get what we want — successive angles at the top of the tape which, as we fold, continue to get closer and closer to $\frac{\pi}{7}$!

You might now like to begin the folding shown in Figure 4 and then repeat the parts 7 through 12 until you notice that the pattern of lines on the tape become more and more regular. Throw away the first part of the tape (say, the first 8 triangles) and use the remaining tape to construct your own 7-gon.

Observe that we may also use this tape, after making suitable secondary fold lines, to construct 14-gons, 28-gons, 56-gons, or, more generally, 2^n7-gons. For example, to make the D^2U^1-tape suitable for constructing 14-gons, all you need to do is bisect, by folding, 14 consecutive angles along the top of the tape that already make an angle of $\frac{\pi}{7}$ with the top edge and then perform the FAT algorithm on those 14 consecutive vertices. To obtain 28-gons, you would repeat the process of inserting the secondary fold lines — but this time on the tape that already has lines that make an angle of

[5] Notice how hopelessly inconvenient degree measure would be for describing our procedure.

[6] It may help the reader to follow the argument if he (or she) pencils in the size of the angles referred to in Figure 4 as she (or he) reads the paragraph.

$\frac{\pi}{14}$ with the top edge of the tape.

From what we have just observed about constructing $2^n 7$-gons from tape suitable for constructing 7-gons we can see that our major problem has been reduced to figuring out how to construct N-gons when N is odd.

It is now natural to ask for which odd N can a procedure like the one we used for the 7-gon be obtained? We will discuss this question in the next part of this article.

Meanwhile, you might like to do the following "Paper-folding Experiments" involving period-1 folding procedures. The intended outcomes of these experiments will appear in the next issue of Parabola, along with Part 2 of this article.

Experiment 1: Take a strip of paper and fold it using the D^1U^1-procedure. That is, repeatedly bisect (just once) the angle that appears at the top of the tape, then the angle that appears at the bottom of the tape. This should eventually produce a string of equilateral triangles. See if you can also fold a FAT triangle with this tape, by executing the FAT algorithm at points equally spaced (sufficiently far apart) along the top edge of the tape.

Experiment 2: Take a strip of paper and fold it using the D^2U^2-procedure. That is, repeatedly bisect twice the angle that appears at the top of the tape, then the angle that appears at the bottom of the tape. Throw away the first 10 triangles and experiment with the remaining strip of paper. For example, try folding on just the 'short lines' — or just the 'long lines'. Try also folding a FAT polygon using the smallest angle on the tape, that is the angle between the longest fold line and an edge of the tape. The number of sides in your FAT polygon will tell you the size of the smallest angle on this tape.

Experiment 3: Try, on the basis of your results for Experiments 1 and 2, to guess what will happen here before you do this experiment. Take a strip of paper and fold it using the D^3U^3-procedure. That is, repeatedly bisect three times the angle that appears at the top of the tape, then that at the bottom of the tape. Throw away the first 10 triangles and experiment with the remaining strip of paper. For example, try folding on just the 'short lines', just the 'medium lines, or just the 'long lines'. Try also folding a FAT polygon using the smallest angle on the tape, that is the angle between the longest fold line and an edge of the tape. The number of sides in your FAT polygon will tell you the size of the smallest angle on this tape. Was your guess correct?

Reference

[1] Hilton, Peter and Jean Pedersen, Build Your Own Polyhedra, Addison Wesley, Menlo Park, CA, 1994.