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A SLICE OF THE PI

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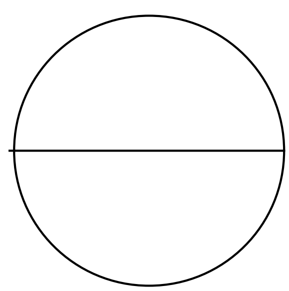
If you were to ask a variety of people what π was, you would probably get a variety of different answers.
The Bible gives π as 3, or at least implies this in chapter 7 of the first Book of Kings verse 23, and this was the value used by the ancient Babylonians.
The ancient Egyptians, in the famous Ahmed Papyrus, (which contains a lot of the mathematics used and developed by the Egyptians) gave a formula for the area of a circle which, if correct, would give π as $\left(\frac{16}{9}\right)^2$, ($\approx 3.1605$).
The Chinese mathematician Chang Hing, writing about 139 AD gives π as $\sqrt{10}$, (≈ 3.16228).
An engineer or surveyor might say $\frac{22}{7}$, which differs from the true value of π in the 3rd decimal place. A better rational approximation, discovered by Archimedes is $\frac{355}{113}$.
In decimal form $\pi \approx 3.141592654.....$ while $\frac{22}{7} = 3.142857...$ and $\frac{355}{113} \approx 3.1415929...$
A teacher of Primary school might (correctly) say, that π is the ratio of the circumference of any circle to its diameter, which gives a lot more insight into what π is than simply giving a numerical value.



A Pure Mathematician might say

$$\pi = 4\int_0^1 \frac{1}{1+x^2}\,dx.$$

(If you haven't done 3 Unit Maths in Year 12, this won't make any sense to you.)
In 1783 William Shanks published the decimal expansion of π to 707 decimal places. It took him 15 years (!) to do the calculation (by hand of course). When modern computers were first being developed, one of the designers' favourite test programs was to compute the value of π. Thus, in 1949, π was calculated to 2035 decimal places and the digits were compared with Shanks' computations. Unfortunately, Shanks made a *mistake* in 528th digit, and so all the digits after that were **wrong**. Shanks would have

been horrified no doubt that so many years of his life had been wasted, but mercifully he was not around to hear the bad news.
There have been many books written which contain a remarkable number of incredible results about this very beautiful number, but as the title of this article implies, I want to just take a 'slice' of some of the many amazing facts about this number.
Several years ago, a newspaper article appeared in America, on April 1st to be more precise, which claimed that 'computer scientists' had 'proven' π to be a rational number, that is, a number which can be expressed as a fraction. The article said that the scientists got a computer to calculate π to a large number of decimal places and found that eventually, they got a very large number of zeros, and hence they concluded that π was rational! As you may have guessed by the date, the article was a practical joke, but embarrassingly an Australian journalist, who happened to see it, reprinted the article in the Science and Technology section of the Australian newspaper several months later! I can still recall a yell of horror from one of my colleagues in the tea room at lunch time when he read it, and he quickly went to 'phone the newspaper to tell them of the impossibility of this nonsense.
One of the interesting properties about the number π is that it is **not** rational, that is, it cannot be written as a fraction. Fractions such as $\frac{22}{7}$ and $\frac{355}{113}$ are merely *approximations* and are not equal to π. The proof of the irrationality of π is quite complicated and involves some intricate use of calculus.
The number π then is certainly irrational but it differs in a certain sense from some of the other examples of irrational numbers such as $\sqrt{2}$. The number $\sqrt{2}$ is a solution of a simple polynomial (in fact quadratic) equation with integer co-efficients, i.e $x^2 - 2 = 0$. The irrational number $\sqrt[3]{2-\sqrt{2}}$ is a solution of $x^6 - 4x^3 + 2 = 0$. Such numbers which are the roots of polynomials with integer co-efficients are called *algebraic* numbers. (Can you prove that $\cos 60°$ and $\cos 20°$ are algebraic?)
The number π is NOT an algebraic number. That is, we cannot find a polynomial equation with integer co-efficients which has π as a solution. This fact was proven in 1882 by Lindemann. The proof, as you might expect is very difficult. If you have studied senior mathematics and have met the number e ($\approx 2.71828..$), you might like to know that e is also not algebraic. By the way, in a very precise sense, *most* real numbers are of this type!
The great mathematician Euler, who lived in the 18th century, did some rather incredible mathematics using infinite series. Some of what he did was in fact incorrect and other parts of it, which gave correct results, were based on some very dubious methods. Nonetheless, Euler did manage to arrive at some remarkable (and correct) conclusions by playing with the infinite. One particularly amazing result he achieved was the following. I must stress that what follows is NOT correct mathematics although the arguments can be made water-tight with a lot of work.
Recall that if I give you a polynomial then there is a simple relationship between the roots of the polynomial and it's co-efficients[1].

[1] See Parabola Vol.33 No. 2 (1997)

Suppose we have a polynomial $p(x)$ with constant term 1,

$$p(x) = 1 + a_1x + a_2x^2 + ... + a_nx^n,$$

and roots $\alpha_1, \alpha_2, ..., \alpha_n$. Then

$$x^n p(\frac{1}{x}) = \left(1 + \frac{a_1}{x} + ... + \frac{a_{n-1}}{x^{n-1}} + \frac{a_n}{x^n}\right) x^n = a_n + ... + a_1x^{n-1} + x^n$$

is a polynomial with roots $\frac{1}{\alpha_1}, \frac{1}{\alpha_2}, ..., \frac{1}{\alpha_n}$ and the sum of the roots is simply $\frac{-a_1}{1} = -a_1$. That is

$$\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + ... + \frac{1}{\alpha_n} = -a_1 \qquad (1).$$

We now look at the function $f(x) = \sin x$. Euler thought of $\sin x$ as a sort of 'polynomial with infinitely many terms'. To find out what the polynomial is, suppose we can write

$$\sin x = a_0 + a_1x + a_2x^2 + ... + a_nx^n + ...$$

Putting $x = 0$ we have $a_0 = 0$. If we differentiate both sides (assuming that we are allowed to differentiate infinitely many terms) we obtain

$$\cos x = a_1 + 2a_2x + 3a_3x^2 + ...$$

and again putting $x = 0$ we have $a_1 = 1$. If we continue to differentiate and put $x = 0$ we find that $a_2 = 0, a_3 = -\frac{1}{6}, a_4 = 0, a_5 = \frac{1}{120}, ...$ (You might like to try this out and see if you can see a pattern.)

Thus, assuming that what we are doing actually makes sense, we have

$$\sin x = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - .....$$

Now, armed with these ideas we can follow Euler's path.

He took the above 'polynomial' for $\sin x$ and wrote

$$\frac{\sin x}{x} = 1 - \frac{1}{6}x^2 + \frac{1}{120}x^4 - .....$$

and observed that the left hand side is 0 precisely when $x = \pm\pi, \pm 2\pi, \pm 3\pi, ...$ so he thought of these numbers as the 'roots' of the polynomial on the right hand side. He then put $w = x^2$ giving

$$\frac{\sin\sqrt{w}}{\sqrt{w}} = 1 - \frac{1}{6}w + \frac{1}{120}w^2 - .....$$

which is a polynomial in w with constant term 1, and with roots $w = \pi^2, 4\pi^2, 9\pi^2, ...$

The final step in his work was to use equation (1), noting that $a_1 = -\frac{1}{6}$ and so we have

$$\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + ... = \frac{1}{6}$$

or

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + ... = \frac{\pi^2}{6}.$$

Most mathematicians are nowadays justly horrified at the way Euler played with the infinite in such a cavalier way (since one can arrive at *incorrect* answers if one is not careful), but in this case Euler did in fact arrive at a correct result, which is one of the most beautiful in all of mathematics. In words it says that the sum of the reciprocals of the squares of the positive integers gets closer and closer to $\frac{\pi^2}{6}$ the more terms we take! For example, if we take a million terms of the series we get approximately 1.644933968, while $\frac{\pi^2}{6}$ is approximately 1.644934068, an error of about 10^{-7}.

Euler went further and did a similar calculation using $\cos x$ instead of $\sin x$ and obtained the equally stunning result

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ... = \frac{\pi^2}{8}.$$

Nowadays these results are obtained in a rigourous way using a branch of mathematics called Fourier Series, which had not been discovered in Euler's time.

Let us now turn from what is commonly known as 'analysis' to geometry.

The ancient Greeks (and others) tried desperately to find a construction which gives a length π. By 'construction' they meant, a finite number of steps using only a pair of compasses and an (unmarked) ruler starting with a line segment of unit length. In particular they attempted to perform a construction known as 'squaring the circle'. The problem was to construct a square with the same area as a given circle. If we make the circle radius 1 then the problem was to construct a square with area π, i.e. with side length $\sqrt{\pi}$. Unfortunately for the ancient Greeks, this cannot be done using only the tools I mentioned above.

On the other hand if infinitely many constructions are allowed, then we can construct a sequence of lengths whose sum approaches π. This was first done, as far as I know, by Vieta (about 1580). What he does is to construct a regular polygon inside a circle, and by making the number of sides larger and larger, our polygon gets closer and closer to the circle. So if the radius of the circle is 1, then the perimeter of the inscribed polygon must be getting closer to 2π. The construction is as follows:

Draw a regular polygon with n sides inside a circle of radius 1.

For the sake of the diagram I have taken $n = 3$ and drawn the 3 sided polygon ACE.

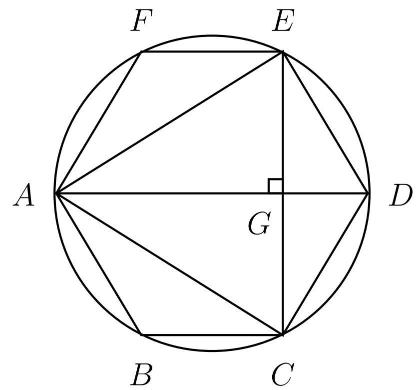
Now bisect each of the sides and draw the regular inscribed polygon with $2n$ sides. (In the diagram this is the hexagon $ABCDEF$.)

Let s_n denote the side length of the polygon with n sides, and so s_{2n} is the length of the side of the polygon with $2n$ sides. Thus, in the diagram below, $|AD| = 2, |CE| = s_n$ and $|DE| = s_{2n}$.

Clearly the triangles EDG and CDG are congruent, (why?) and so EC intersects AD at right-angles. Also recall that the angle $\widehat{AED}$ is a right-angle.

Now the area of $\Delta ADE = \frac{1}{2}|DE|.|AE|$, but also if we think of AD as the base, the area of the triangle is also equal to $\frac{1}{2}|AD|.|EG|$ and so

$$|DE|.|AE| = |AD|.|EG|$$



Using Pythagoras' theorem, we have

$$|DE|\sqrt{|AD|^2 - |DE|^2} = |AD|.\frac{1}{2}|EC|$$

and so, substituting, we get

$$s_{2n}\sqrt{4 - s_{2n}^2} = s_n.$$

Squaring both sides and re-arranging the terms, gives

$$s_{2n}^4 - 4s_{2n}^2 + s_n^2 = 0$$

which is just a quadratic in s_{2n}^2
Solving this equation for s_{2n} and taking the negative root (since $s_n < 2$ and $s_4 = \sqrt{2}$), we have the relationship

$$s_{2n} = \sqrt{2 - \sqrt{4 - s_n^2}}.$$

This tells us how to build up the side length of the $2n$-gon from the side length of the n-gon.

Now $s_4 = \sqrt{2}$ so $s_8 = \sqrt{2 - \sqrt{2}}$, $s_{16} = \sqrt{2 - \sqrt{2 + \sqrt{2}}}$ and $s_{32} = \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2}}}}$. Continuing the process, we have

$$s_{2^n} = \sqrt{2 - \underbrace{\sqrt{2 + \sqrt{2 + \sqrt{2 + \ldots}}}}_{n-2 \text{ times}}}$$

The perimeter of the regular 2^n-gon is thus $2^n s_{2^n}$, and so, as the circumference of the circle is 2π, we have

$$2^{n-1}\sqrt{2 - \underbrace{\sqrt{2 + \sqrt{2 + \sqrt{2 + \ldots}}}}_{n-2 \text{ times}}} \longrightarrow \pi \text{ as } n \to \infty.$$

For example, if we put $n = 6$ in the above formula, we get

$$2^5\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}} \approx 3.140339$$

which is correct to two decimal places. (You might like to experiment with your calculator or computer to see how many terms are needed to give π correct to three decimal places.)

For thousands of years, the number π has fascinated the human mind. It is one of the most interesting numbers in mathematics and has the habit of turning up when least expected.

To finish this brief look at π, let me quote a 'well-known' but impressive formula which links together in a profoundly simple way, the four 'fundamental' constants of mathematics

$$e^{i\pi} = -1.$$

A Large Prime Number

Question What is the largest prime number?
Answer There is no largest prime number (can you prove this?).

Question What is the largest **known** prime number?
Answer The largest known prime number is

$$2^{3021377} - 1.$$

It has 909,526 digits (try writing down a number this big!) and was discovered on 27th January by a 19 year-old student called Roland Clarkson as part of GIMPS (the Great Internet Mersenne Prime Search). All the largest known prime numbers are Mersenne primes (those of the form $2^p - 1$ where p is a prime number), and this is the 37th one found.

For more information on large prime numbers (including Mersenne primes), visit the internet address

`www.utm.edu/research/primes/notes`