Parabola Volume 34, Issue 1 (1998)

SOLUTIONS TO PROBLEMS 1006-1014

Q.1007 A student receives a mark out of 7 for each of the subjects English, Maths and Science. In how many ways can the student get

(a) a total mark of exactly 7;

(b) a total mark of at most 7;

(c) a total mark of exactly 16.

ANS.

(a) The problem is equivalent to finding the number of non-negative integer solutions to

$$x_1 + x_2 + x_3 = 7.$$

To find this, consider $7 + 2 = 9$ ones. Choose any 2 of them and change them to vertical strokes e.g.

$$\begin{array}{ccccccccc} 1 & | & 1 & 1 & 1 & 1 & | & 1 & 1 \\ 1 & & & & 4 & & & 2 & \end{array}$$

This gives a partition of 7 into three parts. The number of ways these 2 ones can be chosen is $\binom{9}{2} = 36.$

(b) Add in an imaginary subject Nonsense in which up to 7 marks can be awarded. A total of at most 7 in E, M and S is equivalent to a mark of exactly 7 in E, M, S and Nonsense, so using the method of (a) we draw $7 + 3 = 10$ ones and change 3 of them in $\binom{10}{3} = 120$ ways.

(c) Here we are finding the number of non-negative integer solutions to

$$x_1 + x_2 + x_3 = 16, \tag{1}$$

with $x_1 \leq 7,\ x_2 \leq 7,\ x_3 \leq 7$. Consider the complement of these inequalities:

$$x_1 \geq 8 \quad \text{or} \quad x_2 \geq 8 \quad \text{or} \quad x_3 \geq 8.$$

If $x_1 \geq 8$, put $y_1 = x_1 - 8$, and equation (1) becomes

$$y_1 + x_2 + x_3 = 8,$$

which has $\binom{10}{2}$ non-negative integer solutions. Similarly for the other cases. Note however that each of the 3 cases

$$x_1 \geq 8 \text{ and } x_2 \geq 8, \quad x_1 \geq 8 \text{ and } x_3 \geq 8, \quad x_2 \geq 8 \text{ and } x_3 \geq 8$$

has been counted twice, all corresponding to $y_1 + y_2 + y_3 = 0$ (which has only 1 solution). Thus the answer is

$$\binom{18}{2} - \left[3\binom{10}{2} - 3\right] = 285.$$

Q. 1008 If n is a positive integer, express n^3 as the sum of n consecutive odd numbers.

ANS. Write

$$\begin{aligned} n^3 &= (k+2) + (k+4) + \cdots + (k+2n) \\ &= \frac{n}{2}[2k + 2n + 2], \end{aligned}$$

with k odd. Solving for k, we have $k = n^2 - n - 1$ (which is always odd) and thus

$$n^3 = (n^2 - n + 1) + (n^2 - n + 3) + \cdots + (n^2 - n + (2n-1)).$$

Q. 1009 If n is an integer, prove that the number $n^4 + 3n^2 + 3$ must have at least one prime factor of the form $4r + 3$.

ANS. Since $n^4 + 3n^2$ is always divisible by 4, $n^4 + 3n^2 + 3$ is of the form $4N + 3$.
Also every odd prime has the form $(4r + 1)$ or $(4r + 3)$. If **all** the prime factors of $n^4 + 3n^3 + 3$ are of the form $(4r + 1)$ then so is their product $n^4 + 3n^2 + 3$, which is not true. Thus $n^4 + 3n^2 + 3$ must have at least one (in fact an odd number) of prime factors of the form $4r + 3$.

Q. 1010 Take any right-angled triangle ABC, right-angled at A. Let T_1, T_2, T_3 be equilateral triangles drawn on the sides AB, AC and BC respectively. Prove that

$$\text{area of } T_3 = \text{area of } T_1 + \text{area of } T_2.$$

ANS. The area of an equilateral triangle with side a is $\frac{\sqrt{3}a^2}{4}$.

$$\begin{aligned} \text{So} \qquad & \text{Area of } T_1 + \text{ area of } T_2 \\ &= \frac{\sqrt{3}}{4}a^2 + \frac{\sqrt{3}}{4}b^2 \\ &= \frac{\sqrt{3}}{4}(a^2 + b^2) = \frac{\sqrt{3}}{4}c^2 \\ &= \text{area of } T_3. \end{aligned}$$

Q. 1011 The only solution to the equations

$$\begin{aligned} x_1 + x_2 &= 0 \\ x_2 + x_3 &= 0 \\ x_3 + x_1 &= 0 \end{aligned}$$

is $x_1 = x_2 = x_3 = 0$, but $x_1 = x_3 = 1, x_2 = x_4 = -1$ is a non-zero solution of

$$\begin{aligned} x_1 + x_2 &= 0 \\ x_2 + x_3 &= 0 \\ x_3 + x_4 &= 0 \\ x_4 + x_1 &= 0. \end{aligned}$$

For what values of r and n do the following equations have a non-zero solution

$$\begin{aligned} x_1 + x_2 + \cdots + x_r &= 0 \\ x_2 + x_3 + \cdots + x_{r+1} &= 0 \\ &\cdots \\ x_{n-r} + x_{n-r+1} + \cdots + x_{n-1} &= 0 \\ x_{n-r+1} + x_{n-r+2} + \cdots + x_n &= 0 \\ x_{n-r+2} + x_{n-r+3} + \cdots + x_1 &= 0 \\ &\cdots \\ x_n + x_1 + \cdots + x_{r-1} &= 0. \end{aligned}$$

ANS. Subtracting the first two equations gives $x_1 = x_{r+1}$. Similarly (if $r < n-1$) from equations $r+1$ and $r+2$, we get $x_{r+1} = x_{2r+1}$, etc.; and from equations $n-r+1$ and $n-r+2$ we get $x_{n-r+1} = x_1$. Hence

$$x_{n-r+1} = x_1 = x_{2r+1} = \cdots$$

and similarly

$$\begin{aligned} x_{n-r+2} = x_2 = x_{r+2} &= x_{2r+2} = \cdots \\ \cdots \quad \cdots \quad &\cdots \\ x_n = x_r = x_{2r} &= x_{3r} = \cdots \end{aligned} \tag{1}$$

(a) If $n = rk+1$ where k is an integer, then n is in the list $1, r+1, 2r+1, \ldots$ and so

$$\begin{aligned} x_1 &= x_{r+1} = x_{2r+1} = \cdots = x_{rk+1} = x_n \\ &= x_r = x_{2r} = \cdots = x_{rk} = x_{n-1} \\ &\cdots \quad \cdots \quad \cdots \\ &= x_3 = x_{r+3} = \cdots = x_{rk-r+3} = x_{n-r+2} \\ &= x_2 = x_{r+2} = \cdots = x_{rk-r+2}. \end{aligned}$$

So the first equation becomes

$$rx_1 = x_1 + x_2 + \cdots + x_r = 0$$

and so $0 = x_1 = x_2 = \cdots = x_r = \cdots = x_n$.

(b)Similarly, if $n = rk+i$ where $i = 1, 2, \ldots$ or $r-1$, then n is in the list $i, r+i, 2r+i, \ldots$. So

$$x_i = x_n = x_r = x_{n-1} = \cdots = x_1$$

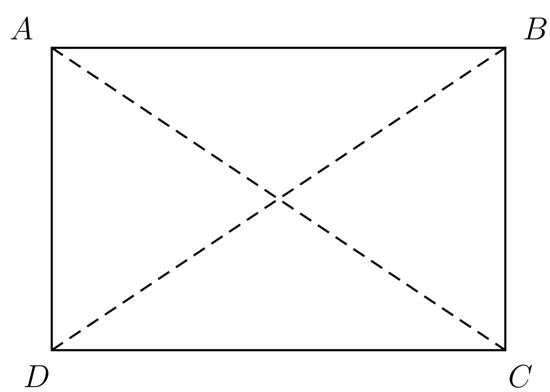
and again $$x_1 = x_2 = \cdots = x_n = 0.$$

(c) If $n = rk$, the equations (1) form r different cycles. All the given equations are satisfied in this case by

$$\begin{aligned}
x_1 &= x_{r+1} = x_{2r+1} = \cdots = x_{n-r+1} = a_1 \\
x_2 &= x_{r+2} = x_{2r+2} = \cdots = x_{n-r+2} = a_2 \\
&\cdots \qquad \cdots \qquad \cdots \\
x_r &= x_{2r} = x_{3r} = \cdots = x_n = -(a_1 + a_2 + \cdots + a_{r-1})
\end{aligned}$$

where $a_1, a_2, \ldots, a_{r-1}$ are **any** numbers.

Q.1012 The following figure shows that it is possible to join every pair of *four* points (no 3 in a straight line) with lines of 2 different colours without drawing a triangle of one of the colours:



What is the largest number of such points?

ANS. The following figure shows that it is also possible with five points:

This is the largest number. For, suppose we had **six** points A, B, C, D, E, F.

From A there are at least 3 lines of one colour (either black or red). Suppose for example that 3 black lines AB, AC and AE leave A. If any one of BC, CE or BE is drawn with black ink, a black triangle results. But if none are black then BCE is a red triangle.

Q.1013 Is it possible for a knight to start at one corner of a chess board, visit every square of the board and end up at the diagonally opposite corner?

ANS. No. Suppose the lower R.H. corner is a white square. If Such a knight's tour were possible, the knight would make 63 moves before reaching the L.H. top corner, another white square. But the colour of the square occupied by the knight changes with each move. After an odd number of moves it must reach a black square.

Q.1014 A sequence $p_1, p_2, \ldots$ is formed as follows:

$$\begin{aligned} p_1 &= 2 \\ p_n &= \text{largest prime number which divides } p_1 p_2 \cdots p_{n-1} + 1 \end{aligned}$$

(For example, $p_2 = 2 + 1 = 3,\ p_3 = 2 \times 3 + 1 = 7$).
What is the smallest prime number which does not occur in this sequence?

ANS. The numbers 2 and 3 occur as the first two terms of the sequence, but 5 never occurs. So the answer is 5. To see that 5 never occurs, we note that if 5 were to occur, we would have

$$p_1 p_2 \cdots p_{n-1} + 1 = 2^r 3^s 5^t.$$

But the left-hand-side is not divisible by 2 or 3, so $r = s = 0$ and

$$p_1 p_2 \cdots p_{n-1} + 1 = 5^t,$$

$$p_1 \cdots p_{n-1} = 5^t - 1.$$

But the right-hand-side is divisible by 4, while the left-hand-side is not.

NOTE. If we changed the question to read

"$p_n =$ **smallest** prime number which divides $p_1 p_2 \cdots p_{n-1} + 1$"

then the problem becomes very difficult (and the answer is not known). See if you can find out whether $5, 7, 11, \ldots$ are in the new sequences. A book prize will be awarded for the best attempt received by 30th September.

Congratulations to Hai Trung Ho who sent in correct solutions to problems 1007(a), 1008, 1009, 1010, 1012 and 1013. Unfortunately Ho did not say what school he or she was from.

Why not get your name into Parabola too? Send in your answers to this issue's problems!