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UNSW SCHOOL MATHEMATICS COMPETITION 1998

SOLUTIONS

JUNIOR DIVISION

1. Let x, y and z be integers. Prove that if $2x + 4y + 5z$ is a multiple of 17, then so is $3x + 6y - z$.

   Solution Write $2x + 4y + 5z = 17k$. Then

$$3x + 6y - z = 10(2x + 4y + 5z) - 17(x + 2y + 3z) = 17(10k - x - 2y - 3z) \ ,$$

   which is a multiple of 17.

2. A regular 21–sided polygon is inscribed in a circle. Is it possible to choose five of its vertices in such a way as to define a pentagon, all of whose sides and diagonals have different lengths?

   Solution Two chords of a circle are of different lengths if and only if the (minor) arcs on which they stand are of different lengths. Consider 21 points equally spaced around a circle, and, for convenience, take the unit of length to be the distance between two adjacent points. Then the possible arc lengths are $1, 2, 3, \ldots, 10$. A pentagon has five sides and five diagonals, so it would appear possible that they are all of different lengths. In fact, if we label the points $0, 1, 2, \ldots, 20$ then by trial and error we can choose $A = 0$, $B = 1$, $C = 4$, $D = 14$, $E = 16$ and the ten lengths are

$$AB = 1\,,\ AC = 4\,,\ AD = 7\,,\ AE = 5\,,\ BC = 3\,,$$
$$BD = 8\,,\ BE = 6\,,\ CD = 10\,,\ CE = 9\,,\ DE = 2\,,$$

   which are indeed all different.

3. A meeting has n invited delegates, Mr. Smith being one of them. At the meeting there is also a journalist who wishes to interview Mr. Smith. The journalist is aware that nobody at the meeting knows Mr. Smith, but Mr. Smith knows everybody. The journalist is allowed to go up to any of the delegates, point to any other delegate, and ask, "Do you know that person?" What is the minimum number of such questions the journalist must ask in order to be sure of identifying Mr. Smith? Prove your answer.

Solution Suppose that the journalist points to a person B and asks person A, "Do you know that person?" Then

- if the answer is "Yes," then B is not Mr. Smith (for nobody knows Mr. Smith), but the journalist cannot tell whether or not A is Mr. Smith;
- if the answer is "No," then A is not Mr. Smith (because Mr. Smith knows everybody), but B remains unknown.

Therefore every question, whatever the answer, serves to eliminate one person. The journalist can thus identify Mr. Smith with $n-1$ questions, but cannot possibly (even if he is lucky!) do so with fewer.

4. Let p, q, r and s be real numbers such that $p^2+q^2=1$ and $r^2+s^2=1$. Prove that

$$(pr+qs)^2 \leq 1 \ .$$

Solution Since $p^2+q^2=1$ and $r^2+s^2=1$, there exist angles α and β such that

$$p=\cos\alpha \ , \ q=\sin\alpha \ , \ r=\cos\beta \quad \text{and} \quad s=\sin\beta \ .$$

Therefore

$$(pr+qs)^2=(\cos\alpha\cos\beta+\sin\alpha\sin\beta)^2=\cos^2(\alpha-\beta)\leq 1 \ .$$

5. Let

$$x = \frac{1}{1998} + \frac{1}{19998} + \frac{1}{199998} + \cdots \ .$$

If $2x$ is written as a decimal, find the 17th digit after the decimal point; also, find the 59th digit after the decimal point.

Solution We have

$$\begin{aligned}
\tfrac{2}{1998} &= \tfrac{1}{999} &&= 0.001001001001001001\cdots \\
\tfrac{2}{19998} &= \tfrac{1}{9999} &&= 0.000100010001000100\cdots \\
\tfrac{2}{199998} &= \tfrac{1}{99999} &&= 0.000010000100001000\cdots
\end{aligned}$$

and so forth; and $2x$ is the sum of all these decimals. Let s_1 be the sum of the digits in the first place after the decimal point in all these numbers, s_2 the sum of the digits in the second place, and so on. Then since the first decimal has a 1 in every third place, the next has a 1 in every fourth, and so on, we see that s_k is equal to the number of factors of k, other than 1 and 2. Therefore $s_{17} = 1$. However, we can not yet deduce that this is the 17th digit of $2x$, as there may be a "carry" from places further to the right. The total in the 18th place, including all carries, is

$$t_{18} = s_{18} + \frac{s_{19}}{10} + \frac{s_{20}}{10^2} + \frac{s_{21}}{10^3} + \cdots \ .$$

We can make the estimates

$$s_{19} < 19 \ , \ s_{20} < 20 < 2 \times 19 \ , \ s_{21} < 21 < 2^2 \times 19 \ , \ \ldots \ ;$$

calculating $s_{18} = 4$ and summing a geometric series, we find that

$$t_{18} < 4 + \left(\frac{19}{10} + \frac{2 \times 19}{10^2} + \frac{2^2 \times 19}{10^3} + \cdots \right) = 4 + \frac{19}{10} \left(1 + \frac{1}{5} + \frac{1}{5^2} + \cdots \right) = \frac{51}{8} \ .$$

This is less than 10, and so there is no carry into the 17th place. Therefore the 17th digit of $2x$ after the decimal point is indeed 1.

Similarly we have $s_{59} = 1$, $s_{60} = 10$ and

$$\begin{aligned} t_{60} &= s_{60} + \frac{s_{61}}{10} + \frac{s_{62}}{10^2} + \frac{s_{63}}{10^3} + \cdots \\ &< 10 + \frac{61}{10} + \frac{62}{10^2} + \frac{63}{10^3} + \cdots \\ &< 10 + \frac{61}{10} + \frac{2 \times 61}{10^2} + \frac{2^2 \times 61}{10^3} + \cdots \\ &= 10 + \frac{61}{10}\left(1 + \frac{1}{5} + \frac{1}{5^2} + \cdots\right) \\ &= 10 + \frac{61}{8}. \end{aligned}$$

So $10 < t_{60} < 20$, and therefore there is a carry of 1 into the 59th place. Hence, the 59th digit after the decimal point is 2.

6. Simplify

$$\frac{2^3-1}{2^3+1}\,\frac{3^3-1}{3^3+1}\,\frac{4^3-1}{4^3+1}\cdots\frac{n^3-1}{n^3+1}\ .$$

Solution We have

$$k^3 - 1 = (k-1)(k^2+k+1) \quad \text{and} \quad k^3+1 = (k+1)(k^2-k+1)\ ;$$

hence

$$k^3 - 1 = (k-1)(k(k+1)+1) \quad \text{and} \quad k^3+1 = (k+1)((k-1)k+1)\ .$$

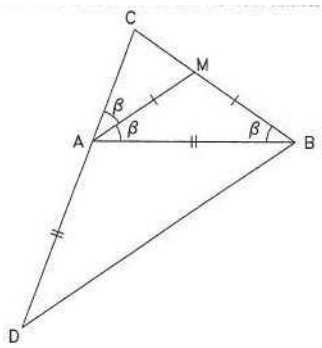
Therefore the numerator of the given product contains the factors $1, 2, 3, \ldots, n-1$ and the denominator contains $3, 4, 5, \ldots, n+1$. Most of these cancel and we are left with $2/n(n+1)$. The numerator also contains factors $2{\times}3+1$, $3{\times}4+1$, $\ldots$, $n(n+1)+1$, and the denominator $1{\times}2+1$, $2{\times}3+1$, $\ldots$, $(n-1)n+1$; again most cancel and there remains $(n(n+1)+1)/(1{\times}2+1)$. Combining all these results gives

$$\frac{2^3-1}{2^3+1}\,\frac{3^3-1}{3^3+1}\,\frac{4^3-1}{4^3+1}\cdots\frac{n^3-1}{n^3+1} = \frac{2}{n(n+1)}\,\frac{n(n+1)+1}{1{\times}2+1} = \frac{2}{3}\,\frac{n^2+n+1}{n^2+n}\ .$$

SENIOR DIVISION

1. In a triangle with sides a, b, c the angle opposite a is twice the angle opposite b. Prove that $a^2 = b(b + c)$.

Solution Extend CA to D such that $AD = AB$, and draw the bisector of angle A, meeting BC at M. Since $\angle CAB$ is twice $\angle CBA$ we can mark angles as shown, and we see that $\triangle AMB$ is isosceles. Looking at exterior angles of triangles ACB and MCA



we have

$$\angle DAB = \angle ACB + \angle CBA \quad \text{and} \quad \angle BMA = \angle MCA + \angle CAM \ .$$

But $\angle ACB$ and $\angle MCA$ are the same angle, and by construction $\angle CBA$ is the same size as $\angle CAM$. Therefore $\angle DAB = \angle BMA$; and since $\triangle DAB$ and $\triangle BMA$ are isosceles, they are similar. Hence $\triangle BDC$ is similar to $\triangle ABC$, because $\angle BDC = \angle ABC$ and $\angle C$ is common to both triangles. It follows that

$$\frac{BC}{CD} = \frac{AC}{CB} \ , \quad \text{that is ,} \quad \frac{a}{b+c} = \frac{b}{a} \ ,$$

and so $a^2 = b(b + c)$.

Alternative solution. Let $\angle B = \beta$, as above. Then $\angle A = 2\beta$ and $\angle C = \pi - 3\beta$. Since $\sin(\pi - \theta) = \sin\theta$ we can apply the sine rule to give

$$\frac{a}{\sin 2\beta} = \frac{b}{\sin\beta} = \frac{c}{\sin 3\beta} \ .$$

Denote each of these three equal ratios by R. Then by using a "sums-to-products" formula and the double-angle formula for sine we have

$$\begin{aligned} a^2 - b(b+c) &= R^2\big(\sin^2 2\beta - (\sin\beta)(\sin\beta + \sin 3\beta)\big) \\ &= R^2\big(\sin^2 2\beta - (\sin\beta)(2\sin 2\beta\cos\beta)\big) \\ &= R^2 \sin 2\beta(\sin 2\beta - 2\sin\beta\cos\beta) \\ &= 0. \end{aligned}$$

Alternative solution. By the cosine rule

$$2ac\cos\beta = a^2 + c^2 - b^2 \ , \qquad (*)$$

and from the sine rule

$$a\sin\beta = b\sin 2\beta \ .$$

Using the double-angle formula for sine and cancelling $\sin\beta$ (which cannot be zero) we have

$$a = 2b\cos\beta \ .$$

Now multiply both sides by ac and substitute from $(*)$ to get

$$a^2c = 2bac\cos\beta = b(a^2 + c^2 - b^2) \ ;$$

collecting all the a^2 terms on the left hand side and factorising both sides gives

$$a^2(c-b) = b(c^2 - b^2) = b(c-b)(b+c) \ .$$

Now if $b \neq c$ then we can cancel $c - b$ to give the required result; while if $b = c$ then $\triangle ABC$ is a right-angled isosceles triangle and we have $a^2 = 2b^2 = b(b+c)$ once again.

2. See Junior Division, question 3.

3. Find all positive numbers x and y such that

$$x^{x+y} = y^{x+2y} \quad \text{and} \quad x^{2x+y} = y^{x+4y} \ .$$

Solution Raise both sides of the first equation to the power $x + 4y$, and both sides of the second to the power $x + 2y$. Then we have

$$x^{(x+y)(x+4y)} = y^{(x+2y)(x+4y)} \quad \text{and} \quad x^{(2x+y)(x+2y)} = y^{(x+4y)(x+2y)} \ ,$$

and hence

$$x^{(x+y)(x+4y)} = x^{(2x+y)(x+2y)} \ .$$

Since x and y are not zero there are now two possibilities: either $x = 1$ or the exponents on the left and right hand sides are equal. If $x = 1$ it is easy to see that $y = 1$. Otherwise,

$$(x+y)(x+4y) = (2x+y)(x+2y) \quad \Rightarrow \quad x^2 = 2y^2 \quad \Rightarrow \quad x = \sqrt{2}\, y \ .$$

Substituting back into the first of the given equations,

$$x^{(1+\sqrt{2})y} = y^{\sqrt{2}\,(1+\sqrt{2})y}$$

and so $x = y^{\sqrt{2}}$. Hence we have

$$y^{\sqrt{2}} = \sqrt{2}\, y \quad \Rightarrow \quad y^{\sqrt{2}-1} = \sqrt{2} \quad \Rightarrow \quad y = \sqrt{2}^{1/(\sqrt{2}-1)} = \sqrt{2}^{\sqrt{2}+1}$$

and $x = \sqrt{2}\, y = \sqrt{2}^{\sqrt{2}+2}$. So the equations have two solutions,

$$x = 1\ ,\ y = 1 \quad \text{and} \quad x = \sqrt{2}^{\sqrt{2}+2}\ ,\ y = \sqrt{2}^{\sqrt{2}+1}\ .$$

4. If k is a positive integer then $k!$ denotes the product of all positive integers up to k: for example, $5! = 1 \times 2 \times 3 \times 4 \times 5 = 120$.

Show that if m and n are positive integers then $(mn)! \geq (m!)^n (n!)^m$.

Solution Splitting the products into blocks of m factors we find that

$$\begin{aligned}(mn)! \ = \ & \big(1 \times 2 \times \cdots \times m\big) \\ & \times \big((m+1) \times (m+2) \times \cdots \times 2m\big) \\ & \times \big((2m+1) \times (2m+2) \times \cdots \times 3m\big) \\ & \times \cdots \\ & \times \big(((n-1)m+1) \times ((n-1)m+2) \times \cdots \times nm\big)\end{aligned}$$

$$
\begin{aligned}
&\geq (1 \times 2 \times \cdots \times m) \times (2 \times 4 \times \cdots \times 2m) \\
&\qquad \times (3 \times 6 \times \cdots \times 3m) \times \cdots \times (n \times 2n \times \cdots \times mn) \\
&= m! \times (2^m \times m!) \times (3^m \times m!) \times \cdots \times (n^m \times m!) \\
&= m! \times m! \times m! \times \cdots \times m! \times (1 \times 2 \times 3 \times \cdots \times n)^m \\
&= (m!)^n (n!)^m .
\end{aligned}
$$

5. See question 5 in the Junior Division.

6. A circle on diameter AB is given, together with a point P inside the circle but not on AB. Show how to construct, using only an unmarked ruler, a line through P perpendicular to AB. Prove that your construction succeeds.

   Solution Draw a chord through A and P, intersecting the circle again at M, and a chord through B and P, intersecting the circle again at N. Let X be the intersection of the lines AN and BM. Since the angle in a semicircle is a right angle, AM and BN are altitudes of $\triangle ABX$. But the three altitudes of a triangle are concurrent, and so PX (extended) is the third altitude of $\triangle ABX$. Therefore PX is perpendicular to AB.