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SOLUTIONS TO PROBLEMS 1015-1024

Q.1015 Quantities of coins are available denominated at one tenth, one twelveth and one sixteenth of a penny. How can these be used to settle a debt of one two hundred and fortieth of a penny? The giving of change is allowed.

ANS. The theoretical point behind problems of this sort is that if m and n are positive integers and d is the greatest common divisor, there exist integers X and Y such that

$$mX + nY = d.$$

There is a process for calculating X, Y given m, n called the extended Euclidean algorithm. The equation $mX + nY = d$ can be rewritten

$$X/n + Y/m = d/(mn) = 1/g,$$

where $g = mn/d$ is the least common multiple of m and n. Thus if m and n are positive integers with least common multiple g, coins with values $1/m$ penny and $1/n$ penny can be used to settle a debt of $1/g$ penny.

We first note that $1/10 - 1/12 = 1/60$ and then note that $4/60 - 1/16 = 1/240$.
Thus $4/10 - 4/12 - 1/16 = 1/240$ as required.

Q. 1016 Show how six cylindrical pencils of equal radius each with neither end sharpened can be put into mutual contact along their curved surfaces.

ANS. The task is to place six (or even seven) equal cylindrical pencils in mutual contact along their curved surfaces. This problem is left open until the next issue is being prepared in the hope that someone will send in a diagram of print quality.

Q. 1017 By considering an equilateral triangle inscribed in a unit circle show that $2\pi > 6\sin 60°$. Next consider a regular 6-gon inscribed in the same circle and show that $2\pi > 12\sin 30° > 6\sin 60°$. Thus show that $2\pi > 192\sin 1.875°$.

Now, using only the square root facility on your pocket calculator and the formula $\sin\theta = \sqrt{\frac{(1-\sqrt{1-\sin^2 2\theta})}{2}}$ which is valid for $0° \le \theta \le 180°$, to find (decimal) fractions r_1, r_2, r_3, r_4, r_5 such that $\sin 30° > r_1, \sin 15° > r_2, \sin 7.5° > r_3, \sin 3.75 > r_4, \sin 1.875° > r_5$ and $96r_5 > 3\frac{10}{71}$. You have now worked out the substance of Archimedes' proof that $\pi > 3\frac{10}{71}$.

Now prove that $3\frac{1}{7} > \pi$ in this style.

ANS. The problem seeks a proof in the style of Archimedes that $\pi < 22/7$. Archimedes considered for $n \ge 2$ a regular $2n$-gon circumscribing a unit circle. The perimeter of the polygon is $4n\tan(90/n)°$, and he took as axiomatic the assertion that this was greater than 2π, the circumference of the circle. Thus, for all $n \ge 2$, $\pi < 2n\tan(90/n)°$. Archimedes decided to calculate $\tan(90/n)°$ for n = 3, 6, 12, 24, 48, 96. The modern reader has the advantage of being able to calculate the following table electronically:

n	$\tan(90/n)°$	$2n\tan(90/n)°$
3	.577 350 27	3.464 102
6	.267 949 19	3.215 390
12	.131 652 50	3.159 600
24	.065 543 46	3.146 086
48	.032 737 61	3.142 715
96	.016 363 92	3.141 873
192	.008 630 96	3.141 670

Archimedes had available only a primitive method of calculating square roots in the standard formula for evaluating $\tan(\theta/2)$ in terms of $\tan\theta$. A proof that $\pi < 22/7$ may now be constructed by working out decimal numbers slightly bigger than each of those in the second column and proving line by line that the required tangent is at most these decimal numbers.

Q.1018 You are given two 20-litre containers full of milk, one empty 5-litre container and one empty 4-litre container. No other measuring container is available. How do you pour milk from one container to another so as to end up with exactly two litres in each small container, 20 litres in one large container and 16 litres in the other?

ANS. It turns out that 9 steps are needed to achieve the desired result: starting with cans of volumes 20, 20, 5 and 4 litres with the first two full and the other two empty to reach a situation in which there is 2 litres in each of the last two. The contents of the cans after each step are as given in the following table:

(0)	20	20	0	0	(5)	14	20	5	1
(1)	15	20	5	0	(6)	14	20	2	4
(2)	15	20	1	4	(7)	18	20	2	0
(3)	19	20	1	0	(8)	14	16	2	4
(4)	19	20	0	1	(9)	20	16	2	2

Q.1019 How can a regular hexagon be dissected by straight line cuts into five pieces which can then be re-assembled to make a square?

ANS. We follow Dudeney's description of how to dissect a regular hexagon into 5 pieces that can be re-assembled to form a square. The diagram is his. Cut the hexagon in half and place the two parts together to form the parallelogram $ABCD$. Continue the line DC to E, making CE equal to the height CF. Then, with the point of your compasses at G, draw the semi-circle DHE and draw the line CH perpendicular to DE. Now CH is the mean proportional between DC and CE, and therefore the side of the required square. From C describe the arc HJ and with the point of your compasses at K describe the semicircle DJC. Draw CJ and DJ. Make JL equal to JC, and complete the square. The rest requires no explanation.

Q.1020 Express the fractions $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}$ and $\frac{1}{9}$ each as the quotient of a 4-digit number by a 5-digit number in which each of the digits 1, 2, 3, 4, 5, 6, 7, 8, 9 occurs in either the numerator or the denominator.

ANS. This problem requires intelligent trial and error to discover that

$$2 \times 6\,729 = 13\,458,\ \ 3 \times 5\,832 = 17\,496,\ \ 4 \times 4\,392 = 17\,568,\ \ 5 \times 2\,769 = 13\,845,$$
$$6 \times 2\,943 = 17\,658,\ \ 7 \times 2\,394 = 16\,758,\ \ 8 \times 3\,187 = 25\,496,\ \ 9 \times 6\,381 = 57\,429.$$

Q.1021 When Mr. Banquo, Mr. Macbeth, Mr. Macduff, Mr. Duncan and Mr. Donalbain all accepted parts in an amateur production of *Macbeth*, complications became inevitable. They were heightened by the stage director's decision to give each of these gentlemen the role of which another one is the namesake. Nor was this all. At the last moment, the stage director decided to change all the roles round; and, when the play was finally put on, none of our Thespians played either the role of which he was the namesake, or the part he has rehearsed.

For example, Macbeth was played by the namesake of the part rehearsed by Mr. Banquo. Macduff was played by the namesake of the part rehearsed by Mr. Donalbain. Mr. Macduff, who had rehearsed the part of Donalbain, played that of Banquo.

Who played the part rehearsed by Mr. Macbeth?

ANS. A somewhat painful elimination process shows that Mr. Macduff played the role rehearsed by Mr. Macbeth.

Q.1022 Are there more ways of paying \$1 using coins of denominations 1c, 2c, 5c, 10c, 20c and 50c than there are of paying \$5 using coins of denominations 5c, 10c, 20c, 50c, \$1 and \$2? Find the precise number of ways in each case.

ANS. This problem asks us to compare the number of ways to pay 100c in coins of values 1c, 2c, 5c, 10c, 20c, 50c with the number of ways to pay 100u in coins of values 1u, 2u, 4u, 10u, 20u, 40u. Here 'u' denotes a unit of 5c. In general, we let $f(n)$ denote the number of ways of paying n in coins of values 1, 2, 5, 10, 20, 50 and $g(n)$ denote the number of ways of paying n in coins of values 1, 2, 4, 10, 20, 40. It is required to (a) calculate $f(100)$ and

$g(100)$ and (b) show that $f(n) < g(n)$ for $n > 3$. For this, we consider the product of 6 geometric series:

$$\begin{aligned} & 1 + f(1)X + f(2)X^2 + f(3)X^3 + \ldots \\ = \; & (1 + X + X^2 + \ldots) \times (1 + X^2 + X^4 + \ldots) \times (1 + X^5 + X^{10} + \ldots) \\ \times \; & (1 + X^{10} + X^{20} + \ldots) \times (1 + X^{20} + X^{40} + \ldots) \times (1 + X^{50} + X^{100} + \ldots) \\ = \; & (1 - X)^{-1}(1 - X^2)^{-1}(1 - X^5)^{-1}(1 - X^{10})^{-1}(1 - X^{20})^{-1}(1 - X^{50})^{-1}. \end{aligned}$$

The point is that each way of paying nc in coins of these values contributes 1 to the term in X^n is the expansion of the product. This explains the first = sign. The second is justified by the formula for a geometric progression. The easiest way to work out values of $f(n)$ and $g(n)$ for various values of n is to let a computer algebra system do the work. Thus we find:

$$f(10) = 11, \; f(100) = 4\,562, \; f(1000) = 103\,119\,380, \; g(10) = 13,$$

$$g(100) = 6\,148, \; g(1000) = 157\,399\,801, \; g(1\,000) = 4\,587\,118\,976.$$

This last number is the number of ways of paying \$50 in current Australian coins! Part (b) is rather harder and may need the partial fraction decomposition of the above products.

Q.1023 Given 12 apparently identical objects of which 11 have the same weight and 1 has a slightly different weight, show that it is possible to use a simple balance three times to determine which one has the different weight and to determine whether its weight is greater than the weight of any of the others.

Now suppose that one has 13 apparently identical objects of which 12 have the same weight and 1 has a slightly different weight. Show that there is no such procedure to determine which one has the different weight and whether its weight is greater than the weight of any of the others.

Finally, let N be such that given N apparently identical objects of which $N - 1$ have the same weight and 1 has a slightly different weight it is possible to use a simple balance four times to determine which one has the different weight and to determine whether its weight is greater than the weight of any of the others. What can be said about N?

ANS. In this question you were asked how to start with 12 apparently identical objects, 11 of which have the same weight and one of which is of slightly different weight and, using a beam balance only three times, determine which has the slightly different weight and whether it is heavier or lighter than any of the others. The plan is as follows. Initially put four on one side of the balance and four on the other. If the two sides balance, place three of the remaining four objects and any three of those that have already been tested on the other side. If the two sides balance this second time, the object of different weight has been identified and it may now be weighed against any other of the objects. If the second weighing results in an imbalance, it is known whether the different object is heavier or lighter and it is known to be one of a trio. In this case, put one of the trio on each side of the balance and then ... You should now work out the strategy if the initial weighing results in an imbalance.

If instead there were 13 such objects, 12 of which have the same weight, any plan would have to begin by

(1) weighing 1 against 1, leaving 11 aside, or

(2) weighing 2 against 2, leaving 9 aside, or

(3) weighing 3 against 3, leaving 7 aside, or

(4) weighing 4 against 4, leaving 5 aside, or

(5) weighing 5 against 5, leaving 3 aside, or

(6) weighing 6 against 6, leaving 1 aside.

But in (1), (2), (3) and (4) there may be an initial balancing resulting in the task of separating 22, 18, 14 or 10 possibilities by experiments with only 9 possible outcomes. This cannot be done. Likewise in cases (5) and (6) there may be an initial imbalance resulting in the task of separating 10 or 12 possibilities by experiments with only 9 possible outcomes. Again this cannot be done. So there is no such plan.

If instead four uses of the balance are allowed, 41 or more such objects would give rise to 82 or more possible outcomes which cannot be separated by experiments with only 81 different outcomes. So N, the number of objects for which there is such a plan using four weighings, must be at most 40. A more detailed analysis can reduce this upper limit somewhat.

Q.1024 A frictionless pulley is affixed from the ceiling and a weightless rope passed through it. An inert object is attached to one end of the rope. On the other side of the pulley a monkey of the same weight holds on to the rope. If the monkey climbs up the rope, what happens to the inert object?

ANS. The problem deals with a monkey climbing a rope which passes over a pulley and has an inert object of the same weight as the monkey on the other end. In fact the vertical distance between the monkey and the inert object remains constant. This may be shown by an argument based on Newton's Laws of Motion. The details are omitted.