Parabola Volume 34, Issue 3 (1998)

SURFING BRACHISTOCHRONES

by

Bruce Henry and Simon Watt[1]

One of the most famous problems in the history of dynamics is the *brachistochrone* problem. The problem is to find the shape of the curve that a particle should follow if it is to slide without friction in the minimum time from a higher point to a lower point (not directly beneath it) under the influence of a uniform gravitational field. This problem was invented as a mind-game for mathematicians by Johann Bernoulli in 1696: *"I, Johann Bernoulli, greet the most clever mathematicians in the world... If someone communicates to me the solution of the proposed problem, I shall the publicly declare him worthy of praise"*. The problem immediately grabbed the attention of the big guns. Correct solutions were obtained independently by Newton, Leibniz, L'Hopital, Johann Bernoulli and his brother Jakob. Newton published his solution anonymously, but Johann Bernoulli recognized it as the master's work, *"Ah, I know the lion by his paw"*[2].

In the calculations below we examine two surfing manoeuvres involving brachistochrones. The first calculation presented shows the fastest path to go down the face of a wave and then to turn back up for a possible re-entry from the peel and the second shows the fastest path for tucking into a barrel. Simple conditions are obtained, in terms of the physical parameters describing the wave, under which it is possible to execute each of these manoeuvres.

[1]Bruce and Simon are members of the Nonlinear Dynamics Group in Applied Mathematics at UNSW. Their research collaborations usually concern the dynamics and statistical mechanics of fractal pattern formation. Here they take time out to go surfing. A special thank you to Mick (Teacher/surfer/poet-Lennox and elsewhere) for his feed back.

[2]Excellent sources of information about the lives and achievements of mathematicians are; *Dictionary of Scientific Biography* edited by C.C. Gillespie (New York: Scribner's, 1971); the MacTutor History of Mathematics WEB site: http://www-history.mcs.st-and.ac.uk/history/

The sorts of waves that surfers tend to ride are generally classified either as; *spilling waves* in which turbulent water appears at the wave crest and spills down the face or *plunging waves* (also referred to as *barrels*) in which the face of the wave steepens until it is vertical and then the crest plunges over and splashes into the base. The analysis here is for waves that are steep enough so that surfers can slide down the face but shallow enough so that the surfable region of the face is essentially planar. We have ignored the motion of the wave itself so that the surfer's motion on the wave is simply regarded as motion under the action of a uniform gravitational field. We have also ignored drag forces so that the surfer's motion is regarded as that of a frictionless point particle.

Physical parameters describing surfing waves are; the height H (immediately before breaking) which is the vertical distance from the crest to the trough; the speed v_W at which the wave advances towards shore; an angle α indicating the slope of the wave face ($\alpha = 90°$ is a vertical face); the peel velocity v_P oriented parallel along the wave crest which is the velocity of the junction between the breaking and non-breaking regions of the wave; and the peel angle β defined by

$$\tan\beta = \frac{v_P}{v_W}. \tag{1}$$

To catch a wave a surfer paddles ahead of the wave in the direction of the wave advance. The wave then comes up behind the surfer and tilts the surfboard down the wave face so that it starts to slide. The surfer catches the wave when the forward component of their velocity due to the combined paddling and gravitational sliding is equal to the wave speed. In each of the calculations below it is assumed that the surfer has caught the wave at an initial time $t = 0$ and an initial position A just ahead of the peel and then the surfer slides from rest with respect to the wave.

Problem 1

In this first problem the surfer attempts to slide down the face of the wave, execute a bottom turn and then carve back up the face to enable a re-entry just ahead of the advancing peel in the fastest time possible [3].

The path is shown schematically as a dashed line in figure 1. The x, y co-ordinates

[3] WEB surf to http://www.msp.com.au/tracks/sequence.html (editorial note, February 2014: this is a dead link) and click on August 1998

identify a co-moving frame of reference advancing toward shore with the wave. The x co-ordinate measures the distance down the face of the wave and the y co-ordinate measures the distance along the face of the wave in the direction of the advancing peel.

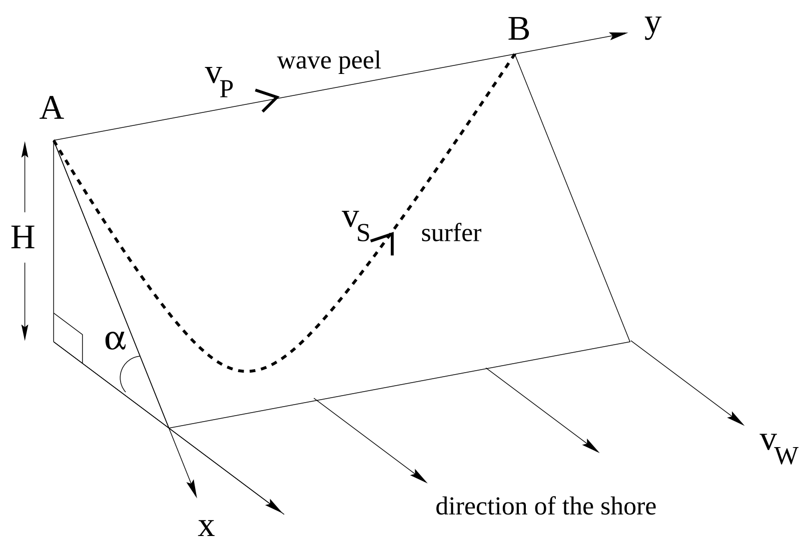

Figure 1: Schematic illustration of a surfer's path (dashed line) for Problem 1.

We now seek the fastest path for the surfer to travel from A to B with the additional constraint that the surfer regains the peel at position B. This is where we turn to the *brachistocrone* problem. The first step in the calculation is to identify the shortest time path from A to B. Let t_S denote the time for the surfer's trip between points x_1 and x_2, then

$$t_S = \int_{x_1}^{x_2} \frac{ds}{v} \tag{2}$$

$$= \int_{x_1}^{x_2} \frac{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{v_S} dx \tag{3}$$

where the surfer's speed v_S at position x down the face of the wave can be calculated using conservation of energy. The gain in kinetic energy of the surfer, $\frac{1}{2}mv_S^2$, is equal to the loss of gravitational potential energy, $mg'x$. Hence

$$v_S = \sqrt{2g'x}. \tag{4}$$

where the gravitational acceleration down the face of the wave, g', is given by

$$g' = g \sin \alpha. \tag{5}$$

Now substitute Eqs.(4), (5) into Eq.(3) and the problem is to find the 'fall-line' $y(x)$ that minimizes

$$t_S = \int_{x_1}^{x_2} \sqrt{\frac{1 + \left(\frac{dy}{dx}\right)^2}{2xg \sin \alpha}} dx. \tag{6}$$

The resulting curve is the *brachistocrone*. We do not enter into the details of this calculation here, which involves methods from the calculus of variations, but simply quote the result found by the masters more than three hundred years ago. The resulting path, a cycloid[4], is defined by

$$x = a(1 - \cos \theta) \tag{7}$$

$$y = a(\theta - \sin \theta) \tag{8}$$

where the constant a is determined by allowing the cycloid to pass through a specified point. The cycloid passes through the deepest point on the wave at $x = 2a, y = \pi a, (\theta = \pi)$ and returns to the top of the wave at $x = 0, y = 2\pi a, (\theta = 2\pi)$ (see Figure 2). This identifies the distance from A to B as $2\pi a$. The time for the peel to advance the distance from A to B is

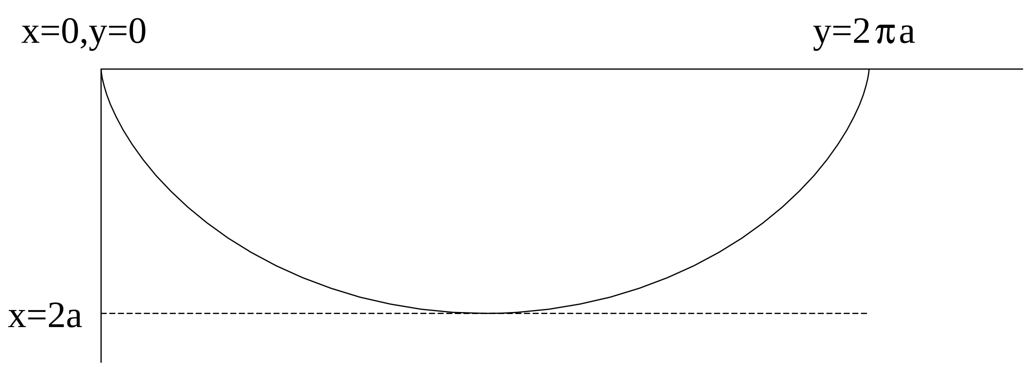

Figure 2: The cycloid represented by Eqs.(7),(8)

thus

$$t_P = \frac{2\pi a}{v_P} \tag{9}$$

The time for the surfer to travel from A to B along the cycloid is

$$t_S = 2 \int_0^{2a} \sqrt{\frac{1 + \left(\frac{dy}{dx}\right)^2}{2xg \sin \alpha}} dx \tag{10}$$

[4]A cycloid is the path traced out by a point on the perimeter of a circle that rolls along a straight line.

Using Eqs.(7),(8) we have, $dx = a \sin\theta d\theta$ and

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{d\theta} \Big/ \frac{dx}{d\theta} &(11)\\ &= \frac{1-\cos\theta}{\sin\theta} &(12) \end{aligned}$$

so that

$$\begin{aligned} t_S &= 2\int_0^{\pi} \sqrt{\frac{1+\left(\frac{1-\cos\theta}{\sin\theta}\right)^2}{2a(1-\cos\theta)g\sin\alpha}}\, a\sin\theta d\theta &(13)\\ &= 2\pi\sqrt{\frac{a}{g\sin\alpha}}. &(14) \end{aligned}$$

The manoeuvre can thus be executed if the shortest time for the surfer to travel from A to B, Eq.(14), is less than the time for the peel to advance from A to B, Eq.(9). We thus require

$$a > \frac{v_b^2}{g\sin\alpha}. \quad (15)$$

The following observations can be made regarding this manoeuvre:

1. The wave height, H, is from simple trigonometry related to a via $H = 2a\sin\alpha$. Hence if the height is less than the critical height

$$H^* = 2\frac{v_P^2}{g}, \quad (16)$$

then the surfer will not be able to regain the peel by following the cycloid and the manoeuvre will not be possible.

2. The faster the wave is peeling, the deeper the surfer needs to go down the wave face before doing a bottom turn to regain the peel.

3. The steeper the wave, the less distance down the wave face the surfer should go before turning back up towards the peel.

As an aside, the surfer's average speed along the cycloid is,

$$v_S = \frac{2}{t_S}\int_0^{2a} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx. \quad (17)$$

Again carrying out the integral along the cycloid, we have

$$v_S = \frac{1}{2\pi}\sqrt{\frac{g\sin\alpha}{a}}2a\int_0^\pi \sqrt{2-2\cos\theta}d\theta \tag{18}$$
$$= \frac{4}{\pi}\sqrt{ga\sin\alpha}. \tag{19}$$

The integral was simplified using the identity $1-\cos\theta = 2\sin^2\frac{\theta}{2}$. This can be compared with the speed of the peel

$$v_P = \frac{2\pi a}{t_S} \tag{20}$$
$$= \sqrt{ga\sin\alpha} \tag{21}$$

Hence the surfer has a faster average speed (by about 27%) by following the cycloid then would be obtained by staying with the peel (if the latter were possible).

Problem 2

In this second problem the surfer takes off from rest (with respect to the wave) at position A and time $t=0$ and follows a path until the surfer's horizontal speed is $v_S = v_P$ in a direction parallel to the crest[5]. From this point on, the surfer continues parallel to the crest. We seek the fastest path for the surfer to achieve this manoeuvre. The manoeuvre is illustrated schematically in Figure 3.

The condition that $v_S = v_P$ requires that the surfer travels a vertical distance

$$x_S = \frac{v_P^2}{2g\sin\alpha} \tag{22}$$

down the face of the wave. This is found by equating the change in gravitational potential energy from the top of the path, $mx_Sg\sin\alpha$ with the increase in kinetic energy from rest to $\frac{1}{2}mv_P^2$. This manoeuvre will not be possible if the wave height is less than the critical height

$$H^* = \frac{v_P^2}{4g}. \tag{23}$$

The above equation, Eq.(23), might quite generally be thought of as the condition for a *close-out*. The fastest path to move from the top of the wave to a point located a vertical

[5]WEB surf to http://www.msp.com.au/tracks/sequence.html (Editorial note, February 2014: this is a dead link) and click on May 1998

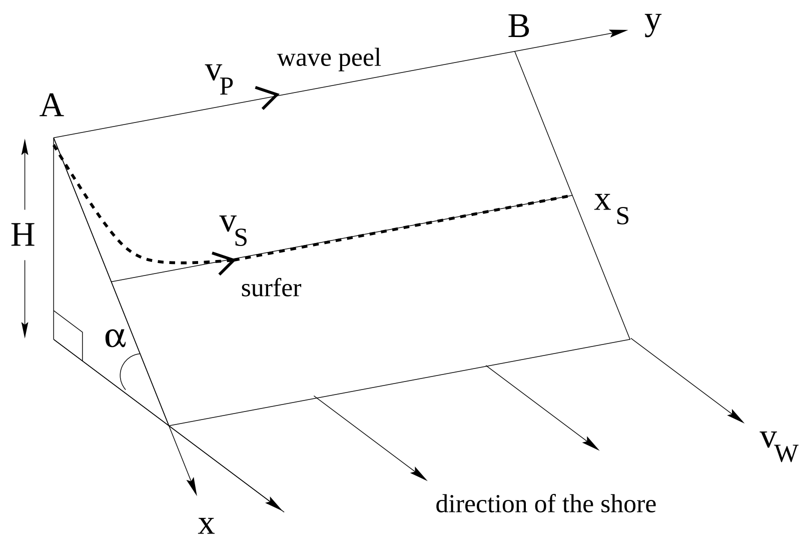

Figure 3: Schematic illustration of a surfer's path (dashed line) for Problem 2.

distance x_S from the top is the straight line free fall path, however this would require a $90°$ turn at the bottom of the drop to keep up with the advancing peel and complete the manoeuvre. The fastest path in general (which allows for horizontal displacement) is the cycloid which starts at A and intersects the line at $x = x_S$ which is parallel to the wave crest. The fastest smooth ride that doesn't require any change in direction at the bottom of the drop is the cycloid passing through $x_S = 2a, y_S = \pi a$. Suppose that the surfer follows this particular cycloid then the time for the surfer to travel to the bottom of the cycloid is

$$t_S = \pi\sqrt{\frac{a}{g\sin\alpha}} \tag{24}$$

It is easy to show that in this manoeuvre the surfer arrives at the bottom of the cycloid before the wave break. The time for the break to arrive at the bottom of the cycloid is the time for the peel to travel a horizontal distance πa plus the time for the break at πa to fall a vertical distance $2a$. Hence

$$t_P = \frac{\pi a}{v_P} + \sqrt{\frac{4a}{g\sin\alpha}}. \tag{25}$$

The second term in the above equation was found by integrating

$$\frac{dx}{dt} = \sqrt{2xg\sin\alpha} \tag{26}$$

from $x = 0$ to $x = 2a$ and then solving for t. Comparing Eqs.(24), (25) we find

$$t_P = \frac{\pi a}{v_P} + \frac{2t_S}{\pi} \tag{27}$$

so that the surfer is ahead of the break. In very steep plunging waves a slower path to the line $x = x_S$ could result in a *tube* ride where the surfer is totally covered up by the wave.

The critical conditions for; i) a slide into a bottom turn allowing a re-entry off the peel, Eq.(16), and ii) a slide into a barrel ride parallel to the crest, Eq.(23), can be related to other physical parameters through empirical relations for typical surfing waves [6]. Typical surfing waves break at a height $H = \frac{3}{4}D$ where D is the depth of the ocean at that location. The typical wave speed of surfing waves is $v_W = \sqrt{gD}$ where g and D are measured in feet per second squared and feet respectively. (Wave heights measured in feet seems to be the surfing standard.) Hence

$$v_w \approx 1.15\sqrt{gH}. \tag{28}$$

Spilling waves typically have steepness $\alpha \approx 30 - 45°$ whereas plunging waves have $\alpha \approx 45 - 90°$. The peel angle might vary from $35 - 80°$ in both types of waves. Intermediate level surfing waves range from about 4 to 10 feet.

The above calculations showed two brachistocrone surfing manoeuvres. The first manoeuvre, the fastest path involving a bottom turn with a possible re-entry off the peel, is well suited to a spilling wave with a low peel speed. The second manoeuvre, the fastest path to tuck into the wall of a wave, might be ideal for making a ride on a steep wave with a high peel speed. Both of these manoeuvres are of course highly idealized. They assume a planar wave face rather than a rounded wave face, they neglect the effects of wave surge, they neglect the effects of drag, and they assume surfers are point particles which they certainly are not[7]. One of the most important idealizations is that we have ignored the effect of the wave motion on the surfer. This motion provides the major difference between the dynamics of skiing[8] and surfing.

[6]One of the most comprehensive studies of surfing waves is *Recreational Surfing Parameters* by J.R. Walker, Technical Report No. 30, University of Hawaii - Look Lab - 73 -30 (1974).

[7]Happy 40th Chunky!

[8]In ski racing, the objective is to minimize the time of descent through a course delineated by poles down a slope. The optimal path between poles is the *brachistocrone*. See for example *The Physics of Skiing* by D.

Ultimately there is no better calculation of the optimal surfing manoeuvre under actual physical conditions then the real time solutions of real world surfers. In this connection it is interesting to note that the manoeuvre described in Problem 1 would require a tight 180° turn or *snap* at the lip if the surfer was to continue on a similar manoeuvre with a re-entry. This sort of tight turn has only recently (within the last decade), through the expertise of world champions like Tom Carroll and Kelly Slater, become a feature of modern surfing[9]. Finally, in figure 4, we compare Ross Clarke-Jones' path as he arcs down the face of a monster wave at Phantom Reef (*Surfing Life* 1999 Big Wave Annual pp3,4, photo by Jeff Hornbaker) with a portion of a cycloid (shown as a solid line). A solution worthy of praise.

Lind and S.P. Sanders (New York: Springer Verlag, 1996).

[9]See for example the description of the *snap* manoeuvre by Kalani Robb in *Surfer* Vol.38 No.4 (1997) p50.

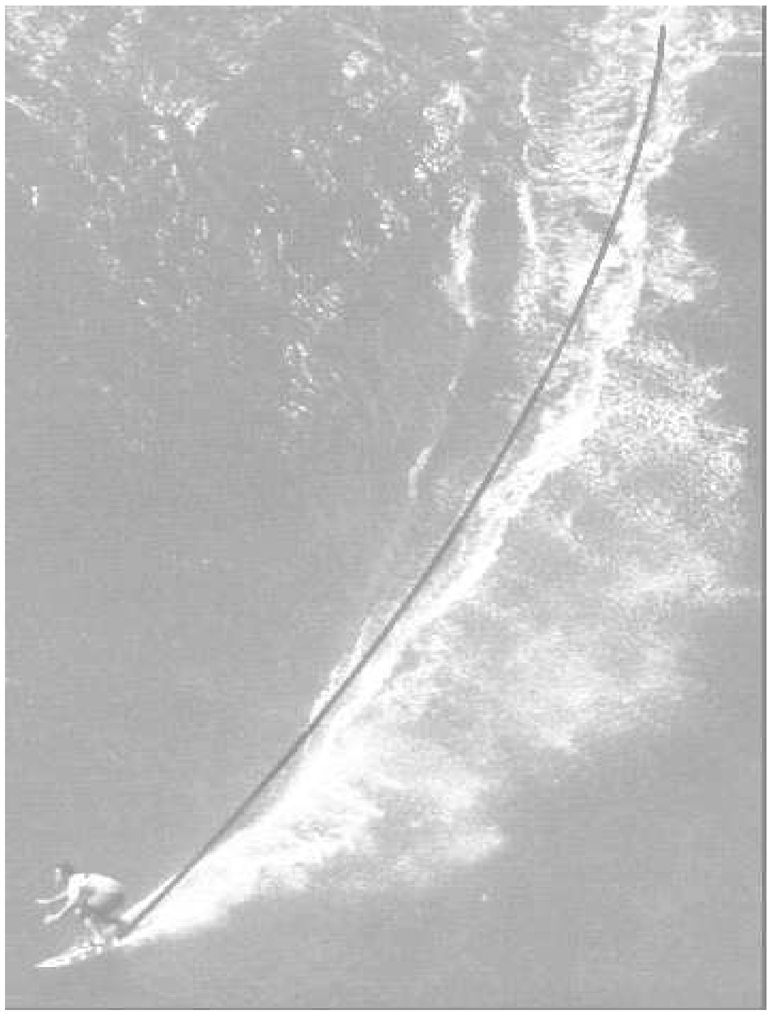

Figure 4: Ross Clarke-Jones' solving Bernoulli's brachistochrone problem.