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COLOURING THE TORUS

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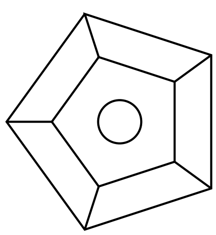
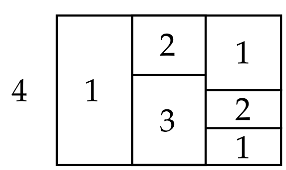
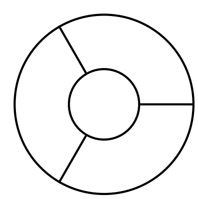
Suppose a certain lover of donuts (we will call him Homer), wants to put coloured icing on his donuts. Homer insists that each region of the donut is coloured in such a way that two regions that are next to each other have different colours. How many colours of icing will Homer need so that he can do this however he divides up the surface of the donut? Suppose one day he finds a "double donut" (with two holes). Now how many colours will Homer need?

Homer's problem may remind you of one of the most famous results in mathematics: the **Four Colour Theorem**, which says that *Any map on a plane or sphere can be coloured by at most four colours*. A map in this sense is just a division of a surface into non overlapping regions: we want adjacent regions to have a different colour. We want to know how many colours we would need if the world were a torus, which is the mathematical name for the surface of a donut (or a bagel if you do not have a sweet tooth).

Now, it is easy to draw maps on the plane which need at least four colours, for example:







The map on the left actually occurs in France with the departments around Paris; note that the map on the right has only four regions. The map in the middle is equivalent to a map of the mainland states of Australia (girt by sea).

Now by "colouring a map" we mean to assign a colour to each of region of the map so that two regions that share a boundary (i.e. an edge) have different colours. As far as this colouring goes, the actual shape of the regions in more or less irrelevant: all that matters is whether they are adjacent or not. So we can restrict ourselves to looking at maps where the boundaries are straight lines.

In *Parabola* vol. 32 no. 1 Peter Brown discussed the Four Colour Theorem, its history and the only known proof, which was found in 1976 by Kenneth Appel and Wolfgang Haken of the University of Illinois (which institution was so impressed by it that

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they used the statement to frank outgoing mail). Appel and Haken performed an exhaustive search of 1936 cases, using a computer. Peter Brown discussed this proof and proved the "6 colour theorem" (6 colours suffice for the sphere or plane). In this article we will look at the case for more complicated surfaces, such as the torus and pretzels (tori with several holes). The corresponding results turn out to be rather easier.

It is relatively easy to find "maps" on the torus that need more than 4 colours. For example consider this:

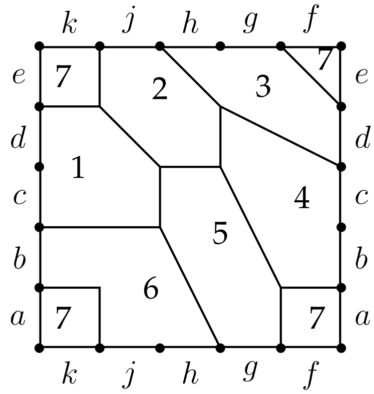


Figure 1

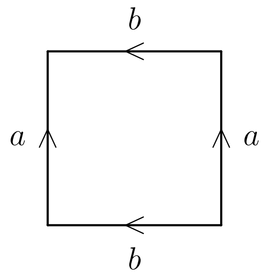
To get the torus, wrap this square up into a cylinder by gluing the vertical edges, and then gluing the ends of the cylinder together to close up the torus. So the edges labelled a are glued together, and similarly for b, c ... Edges a, e, f and k must then be deleted so that the four regions labelled 7 become one region.

If you look at figure 1, you can see that each region touches every other one somewhere. For example, region 1 clearly touches regions 7, 2, 5 and 6. When edges d and c are glued, it also touches region 3 and 4 respectively. You should check the others.

Suppose we have a map on a surface where the map has F countries (F stands for faces). Suppose also that there are V vertices and E edges. The number $F - E + V$ is called the **Euler characteristic**; it is usually written χ (the Greek letter "chi"). In the mainland Australia example above we have 7 faces (including the "sea" surrounding the map), 19 edges and 14 vertices, so $\chi = 2$. Note that we have to consider every junction of two boundaries as a vertex, and that the number of edges is not the same as the number of lines. Any map on the plane (or the sphere) will always have Euler number 2 — try it (but be sure to count all vertices and edges).

For other surfaces, for example the torus, we can similarly define an Euler number as $V - E + F$, but we have to be more careful about what we allow as a map. I will not go into details here, but just say that the great French mathematician Henri Poincaré proved in 1898 that this number depends only on the surface. Furthermore, it stays the same even if we bend the surface, as long as we do not break it. So the cube (for example) has the same Euler number as the sphere i.e. 2.

To work out χ for a torus, consider the following diagram:



We get the torus by gluing the edges labelled a together so the arrows point the same way and then gluing the edges labelled b similarly. We see we have 1 face, 2 edges and, as the four vertices of the square all get matched up in the gluing, only 1 vertex. Hence for the torus $\chi = 1 - 2 + 1 = 0$. We can check this out for the map of figure 1 where we have 10 faces, 26 edges and 16 vertices (note that several vertices are the same point after gluing, for example the four corners of the square come together and so are the same vertex).

We can similarly show that $\chi = 2(1 - g)$ for a "g holed torus" — in fact for any surface $\chi \leq 2$ and we can also prove the result:

For any map on a surface of Euler characteristic χ

$$\frac{2E}{F} \leq 6\left(1 - \frac{\chi}{F}\right). \tag{1}$$

To prove this we need to be exact about what we allow as a map, so I'll omit

Now we can go to the colouring theorems. Our first result is the **Basic Colouring Lemma**:

Suppose that $1 \leq 2E/F < N$ *for every map on a surface. Then every map can be coloured with* N *colours.*

This is proved by induction on the number of faces. Clearly, if there is one face, the result holds, as $N > 1$. Suppose any map with k faces can be coloured with N colours.

Consider a map with $k + 1$ faces. Since $2E/F < N$ the average number of edges per face is less than N and so there is some face with less than N edges. Shrink this face to a point. Then the map we get has N faces and so can be coloured with N colours. But returning to the original map, the face we shrank had less than N edges and so at least one colour does not appear among its adjacent faces. Use this colour to colour the face, and we have the result for $k + 1$. The theorem follows by induction.

In fact, this is the proof Peter Brown gave in the article referred to above for the 6 colour theorem. That result now follows very easily, since $\chi = 2$ for a sphere so $2E/F < 6$.

Now we proceed to our result, which was proved by P.J. Heawood in 1890. Firstly, for a surface with Euler characteristic χ, we define the **Heawood number** as

$$N_\chi = \left\lfloor \frac{7 + \sqrt{49 - 24\chi}}{2} \right\rfloor$$

($\lfloor x \rfloor$ is the greatest integer not exceeding x; so $\lfloor \pi \rfloor = 3$ for example.) For various values of χ we get

χ	2	0	−2	−4	−6	−8	−10	−12
N_χ	4	7	8	9	10	11	12	12

Now we can prove **Heawood's Theorem**, which is the result that answers Homer's original question:

For any surface with $\chi \leq 0$, N_χ colours will colour any map.

We begin by assuming that $F \geq N_\chi + 1$, otherwise the result is very easy: just colour each region with a different colour. Then as $\chi \leq 0$,

$$-\chi/F \leq -\chi/(N_\chi + 1). \tag{2}$$

Now $$N_\chi + 1 > \frac{7 + \sqrt{49 - 24\chi}}{2}.$$

So $$2N_\chi - 5 > \sqrt{49 - 24\chi}.$$

Square and divide by 4 $$N_\chi^2 - 5N_\chi - 6 > -6\chi,$$

or $$N_\chi(N_\chi + 1) > 6(N_\chi + 1 - \chi).$$

So $$N_\chi > 6\left(1 - \frac{\chi}{N_\chi + 1}\right) \geq 6\left(1 - \frac{\chi}{F}\right),$$

using equation (2). So $N_\chi > 2E/F$ from equation (1) and the result follows from the basic colouring lemma.

This theorem shows that N_χ colours will suffice: it doesn't tell us that they are necessary. The map on the torus given earlier is due to Heawood and shows that not only are 7 colours sufficient — which is the theorem — but they are also necessary. In fact, it turns out that for all the surfaces topologically the same as our multi-holed tori, the Heawood number is the minimum number needed. This fact was proved by Gerhard Ringel and J.W.T. Youngs in 1968.

Finally, we note that for the case of the sphere, $N_2 = 4$, and so the Appel and Haken's proof of the Four Colour Theorem means Heawood's Theorem is also true for $\chi = 2$.