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DIFFERENTIATING THE NON-DIFFERENTIABLE — FRACTIONAL CALCULUS

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Introduction

Many of you have now learnt how to calculate the first derivative $\frac{df}{dx}$ for a wide range of functions such as $f(x) = x^{1/2}, f(x) = \sin(x), f(x) = 1$, etc. You also would have learnt how to calculate the second derivative, $\frac{d^2f}{dx^2}$, the third derivative, $\frac{d^3f}{dx^3}$, and so on. You have even learnt how to calculate negative derivatives in the sense that $\frac{d^{-1}f}{dx^{-1}}$ represents one integration of the function with respect to x and $\frac{d^{-2}f}{dx^{-2}}$ represents two integrations with respect to x etc. But have you ever wondered about a fractional derivative? What about $\frac{d^{1/2}f}{dx^{1/2}}$ or $\frac{d^{-1/2}f}{dx^{-1/2}}$ or $\frac{d^qf}{dx^q}$ where q is any number?

Figure 1: The rugged surface of a malignant breast cell nucleus is typical of surfaces that cannot be properly understood using the ordinary calculus but may be amenable to studies using fractional calculus. This image, which was obtained by Andrew Einstein, Mount Sinai School of Medicine, appeared in the 1998 Annual Publication of the American Institute of Physics.

Whether or not you have entertained wonderings about fractional calculus you may be interested to know that the architects of calculus, Newton and Leibniz, had already thought about such things right back in the early days of the development of

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the ordinary (whole number) calculus. In a letter to L'Hopital in 1695, Leibniz discussed the meaning of $\frac{d^{1/2}x}{dx^{1/2}}$ and suggested that the result was "an apparent paradox, from which one day useful consequences will be drawn." In the ensuing three hundred years since then the physical and natural sciences have been filled with applications of the ordinary calculus but few if any "useful consequences" have been drawn from the fractional calculus. Recently, however, some interesting new mathematical discoveries have been made which hold the promise that Leibniz's prophecy on the fractional calculus will soon be realized.

Definitions and Examples

So what is a fractional derivative or a fractional integral? First we introduce notation for ordinary differentiation and integration. An $n-$fold derivative, or differentiation n times is represented by

$$\frac{d^n f(x)}{dx^n} = \frac{d}{dx}\left(\frac{d}{dx}\left(\frac{d}{dx}\cdots\left(\frac{d}{dx}f(x)\right)\right)\right). \tag{0.1}$$

Similarly we have an $n-$fold integral, or integration n times represented by

$$\frac{d^{-n} f(x)}{dx^{-n}} = \int_0^x \left(\int_0^{x_{n-1}} \cdots \left(\int_0^{x_2}\left(\int_0^{x_1} f(x_0)dx_0\right)dx_1\right)\cdots\right)dx_{n-1}. \tag{0.2}$$

(This notation suggests that we think of integration as differentiation, but a whole negative number of times.)

Example

Consider the function $f(x) = \sqrt{x}$.

1.

$$\begin{aligned} \frac{d^2 f(x)}{dx^2} &= \frac{d}{dx}\left(\frac{d}{dx}\left(\sqrt{x}\right)\right) \\ &= \frac{d}{dx}\left(\frac{1}{2}\frac{1}{x^{1/2}}\right) \\ &= -\frac{1}{4}\frac{1}{x^{3/2}} \end{aligned}$$

2.

$$\begin{aligned} \frac{d^{-2} f(x)}{dx^{-2}} &= \int_0^x \left(\int_0^{x_1} \sqrt{x_0}dx_0\right)dx_1 \\ &= \int_0^x [\frac{2}{3}x_0^{3/2}]_0^{x_1} dx_1 \\ &= \frac{2}{3}\int_0^x x_1^{3/2}dx_1 \\ &= 2[\frac{4}{15}x_1^{5/2}]_0^x \\ &= \frac{4}{15}x^{5/2} \end{aligned}$$

A standard result of multiple integration is that the $n-$fold integral appearing in equation (0.2) can equivalently be represented by the single integral

$$\frac{d^{-n}f(x)}{dx^{-n}} = \frac{1}{\Gamma(n)} \int_0^x \frac{f(y)}{(x-y)^{-n+1}} dy \tag{0.3}$$

where $\Gamma(n)$ is a function of n called the gamma function. This function is itself defined by an integral

$$\Gamma(q) = \int_0^\infty y^{q-1} \exp(-y)\, dy; \quad q > 0.$$

Note that this integral exists for both integer and non-integer values for q. (It is easy to verify that $\Gamma(1) = 1$ and not so easy to verify that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. Moreover using integration by parts, it is possible to prove that for any real number q,

$$\Gamma(q+1) = q\Gamma(q)$$

In particular, if q is an integer then $\Gamma(q+1) = q!$.)

It is beyond the scope of this article to prove the compact formula, Eq. (0.3), for $n-$fold integrals, [2] however it is a straightforward exercise to verify that application of the formula gives the correct results for the $n-$fold integrals calculated in the examples above.

Example

Consider the function $f(x) = \sqrt{x}$.

$$\begin{aligned}
\frac{d^{-2}f(x)}{dx^{-2}} &= \frac{1}{\Gamma(2)} \int_0^x (x-y)\sqrt{y} dy \\
&= \int_0^x xy^{1/2} - y^{3/2} dy \\
&= [\frac{2}{3}xy^{3/2} - \frac{2}{5}y^{5/2}]_0^x \\
&= 2[\frac{4}{15}x_1^{5/2}]_0^x \\
&= \frac{4}{15}x^{5/2}
\end{aligned}$$

which agrees with the result obtained previously by integrating $\sqrt{x}$ twice with respect to x.

The formula for $n-$fold integrals in Eq. (0.3) holds the key to defining a fractional integral. The idea is simply to replace the integer n appearing in Eq. (0.3) by a real number q. The right hand side of the equation is still well defined when n is replaced by q in this way. Indeed this is precisely the definition of the fractional integral introduced by Riemann and Liouville.

[2] The proof is a popular assignment question in undergraduate mathematics courses on multiple integration.

Definition - Fractional Integral[3]

Let $q > 0$ denote a real number and f a continuous function. The *fractional integral of f of order $-q$* is given by

$$\frac{d^{-q}f(x)}{dx^{-q}} = \frac{1}{\Gamma(q)} \int_0^x \frac{f(y)}{(x-y)^{-q+1}}\, dy \tag{0.4}$$

The fractional derivative is now defined by applying differentiation a whole number of times to a fractional integral.

Definition - Fractional Derivative

Let $q > 0$ denote a real number and n the smallest integer exceeding q. The *fractional derivative of f of order q* is given by:

$$\frac{d^{q}f(x)}{dx^{q}} = \frac{d^n}{dx^n}\left(\frac{d^{-(n-q)}f(x)}{dx^{-(n-q)}}\right) \tag{0.5}$$

Example

Consider the fractional derivative $\frac{d^{1/2}f}{dx^{1/2}}$ of $f(x) = \sqrt{x}$. In this example, $q = 1/2$, and $n = 1$ is the smallest integer exceeding q. Hence

$$\frac{d^{1/2}f}{dx^{1/2}} = \frac{d}{dx}\left(\frac{d^{-1/2}f}{dx^{-1/2}}\right)$$

where, from Eq. (0.4) with $q = 1/2$,

$$\begin{aligned} \frac{d^{-1/2}f}{dx^{-1/2}} &= \frac{1}{\Gamma(\frac{1}{2})} \int_0^x \frac{\sqrt{y}}{(x-y)^{-\frac{1}{2}+1}} dy \\ &= \frac{1}{\Gamma(\frac{1}{2})} \int_0^x \frac{y}{\sqrt{xy-y^2}} dy. \end{aligned}$$

To simplify the above integral first complete the square in the denominator, then

$$\frac{d^{-1/2}f}{dx^{-1/2}} = \frac{1}{\Gamma(\frac{1}{2})} \int_0^x \frac{y}{\sqrt{\frac{x^2}{4} - (y-\frac{x}{2})^2}} dy.$$

Now consider the change of variables from y to θ via

$$\begin{aligned} y &= \frac{x}{2} + \frac{x}{2}\sin\theta \\ dy &= \frac{x}{2}\cos\theta d\theta \\ y = 0 &\Rightarrow \theta = -\pi/2 \\ y = x &\Rightarrow \theta = \pi/2. \end{aligned}$$

[3] More generally, in the Riemann-Liouville definition the fractional integral is defined with respect to a lower non-zero integration limit.

With this change of variables we can write

$$\frac{d^{-1/2}f}{dx^{-1/2}} = \frac{1}{\Gamma(\frac{1}{2})}\int_{-\pi/2}^{\pi/2} \frac{\frac{x}{2}\cos\theta}{\sqrt{\frac{x^2}{4} - \frac{x^2}{4}\sin^2\theta}}(\frac{x}{2} + \frac{x}{2}\sin\theta)d\theta.$$

To simplify this integral further use the trigonometric identity

$$\cos^2\theta + \sin^2\theta = 1,$$

to cancel out the denominator. Then

$$\begin{aligned} \frac{d^{-1/2}f}{dx^{-1/2}} &= \frac{1}{\Gamma(\frac{1}{2})}\int_{-\pi/2}^{\pi/2}(\frac{x}{2} + \frac{x}{2}\sin\theta)d\theta \\ &= \frac{1}{\Gamma(\frac{1}{2})}[\frac{x}{2}\theta - \frac{x}{2}\cos\theta]_{-\pi/2}^{\pi/2} \\ &= \frac{\pi x}{2\Gamma(\frac{1}{2})} \\ &= \frac{\sqrt{\pi}x}{2}. \end{aligned}$$

Finally the fractional derivative of $\sqrt{x}$ of order one half is

$$\begin{aligned} \frac{d^{1/2}\sqrt{x}}{dx^{1/2}} &= \frac{d}{dx}\left(\frac{d^{-1/2}\sqrt{x}}{dx^{-1/2}}\right) \\ &= \frac{d}{dx}\left(\frac{\sqrt{\pi}x}{2}\right) \\ &= \frac{\sqrt{\pi}}{2}. \end{aligned}$$

Applications

There have been two recent mathematical discoveries that have helped to unlock the power of the fractional derivative. One such discovery is that of fractal functions. Most of the functions that you are familiar with are smooth. This means that locally they can be approximated by a straight line segment. For example the function $f(x) = x^2$ is well approximated by $2x - 1$ at the point $x = 1$ (see figure 2). The derivative of the function at a particular point provides the slope of the straight line approximation or tangent to the curve. As a second example consider the sum of six cosine functions,

$$f(x) = \sum_{n=0}^{5}(\frac{1}{2})^n\cos(3^n x).$$

This function appears highly irregular near $x = 1$ in figure 3(a) but under increasing magnification the function appears smoother and in figure 3(c) we see that it is well

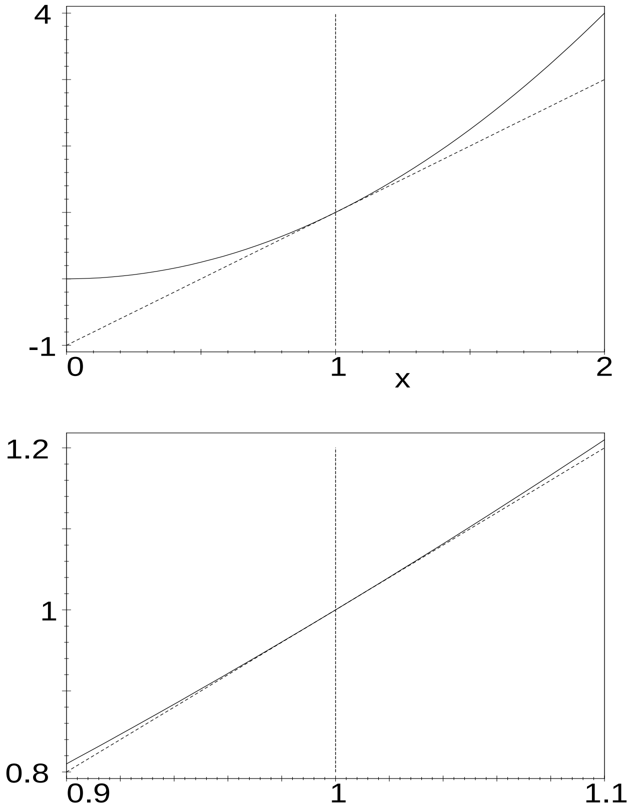

Figure 2: $f(x) = x^2$ with the tangent at $x = 1$. This smooth function can be approximated locally by straight lines.

approximated by a straight line near $x = 1$. Again the derivative of the function evaluated at $x = 1$,

$$f'(1) = -\sum_{n=0}^{5}(\frac{3}{2})^n \sin(3^n) \approx 5.5,$$

is the slope of the straight line approximation at this point.

Fractal functions are not smooth. They have details on all scales and they cannot be approximated locally by straight line segments. An example is the Weierstrass function which can be written as the infinite sum of cosine functions,

$$f(x) = \sum_{n=0}^{\infty}(\frac{1}{2})^n \cos(3^n x).$$

For this function we can see (figure 4) that however closely we zoom in on a point, $x = 1$ say, we continue to find more detail and the slope of the tangent changes orientation under increasing magnification.

Functions such as the Weierstrass function cannot be differentiated (a whole number of times). But it turns out that these fractal functions can be differentiated a fractional number of times, and the fractional calculus is important for studying these differentiability properties. Fractals are characterized by scaling laws and the fractional

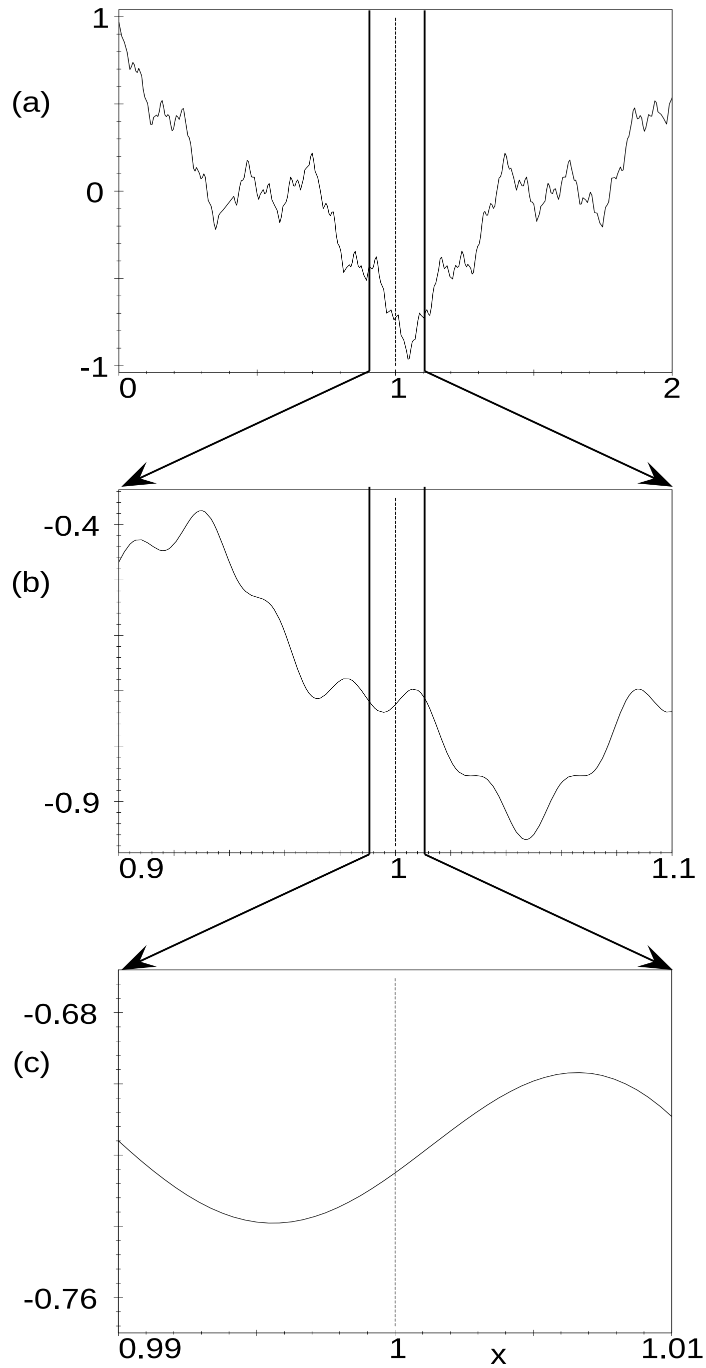

Figure 3: The sum of six cosine functions can be locally approximated by a straight line. The three figures show progressive enlargements of the function near the point $x = 1$. The function is smoother under successive enlargements and the slope of the tangent becomes apparent.

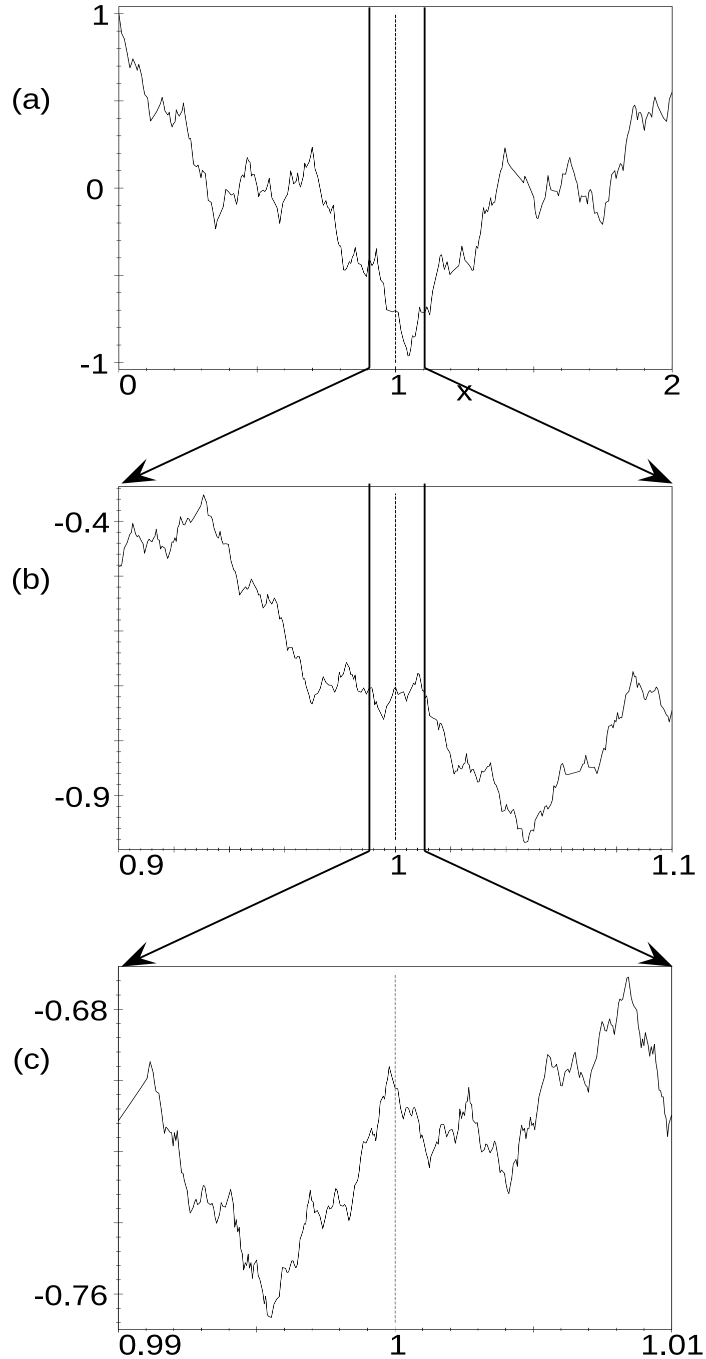

Figure 4: The Weierstrass function cannot be locally approximated by straight line segments. The three figures show progressive enlargements of the function near the point $x = 1$. Roughness occurs on all scales. The slope of the tangent to the curve at $x = 1$ appears to be horizontal in (a), indeterminate in (b) and negative in (c).

derivative at a point can reveal this law. In recent research, scientists at the Mount Sinai School of Medicine have shown that the surfaces of breast cells are fractals and they have found clear differences in the scaling laws for benign cells and malignant cells. The different scaling laws have enabled accurate diagnosis of breast cancers.

The second important new discovery that has brought fractional calculus into prominence is that many physical processes are modelled by fractional differential equations. The importance of a mathematical model is that it can be used to make predictions and to give insight into the physical process that underlies the behaviour. One area where mathematical models have been employed extensively is that of diffusion and transport processes. For example the dispersion of pollutants in the ocean and the motion of electronic charges in conductors are diffusion processes. Here, a probabilistic description leads to a (whole number) differential equation which can be solved to predict average properties of the system. Similar types of equations are used by financial analysts to model stock prices. It has recently been discovered that processes governed by diffusion which is enhanced or hindered in some fashion are better modelled by fractional differential equations than by integer order differential equations. These fractional differential equations are finding numerous applications in areas ranging from financial mathematics to ocean-atmosphere dynamics to mathematical biology.