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UNSW SCHOOL MATHEMATICS COMPETITION 1999

SOLUTIONS

JUNIOR DIVISION

1. See Senior (1).

2. Let $a,\ b$ be the sides and c the hypotenuse of a right–angled triangle. If $a,\ b$ and c are integers, show that
(i) at least one of $a,\ b$ and c is divisible by 5,
(ii) if none of a, b, c is divisible by 7, then either $a+b$ or $a-b$ is divisible by 7.

   Solution (i) If $a = 5n \pm 2$, then $a^2 = 25n^2 \pm 20n + 4$ and so $a^2 - 4$ is divisible by 5. We denote this by writing $a^2 \equiv 4 \pmod 5$. Similarly, all squares are $0, 1$ or $4 \pmod 5$.

   If $a^2 + b^2 = c^2$ and $a,\ b \not\equiv 0 \pmod 5$ then $a^2 \equiv 1,\ b^2 \equiv 4$ or *vice versa*, so $c^2 \equiv 0$ and $c \equiv 0 \pmod 5$.

   (ii) Again, squares are 0, 1, 2 or $4 \pmod 7$.
If $a^2 + b^2 = c^2$ and $a,\ b,\ c \not\equiv 0 \pmod 7$ then $a^2 \equiv b^2 \equiv 1,\ 2$ or $4 \pmod 7$, with $c^2 \equiv 2,\ 4$ or $1 \pmod 7$ respectively. In each case, $a \equiv \pm b \pmod 7$, i.e. 7 divides $(a \pm b)$. ■

3. On a 6×6 chessboard there is a chesspiece (a king) placed in the lower left hand corner. In how many different ways can he reach the top right hand corner if he is never allowed to move "backwards", that is, he can move to a neighbouring square either upwards or to the right or diagonally upwards to the right?

   Solution You can put a number in each square of the chessboard equal to the number of ways of getting to that square. It is equal to the sum of the numbers in the squares to the left, below, and diagonally below to the left. You start by putting 1 in the bottom left–most square. In this way you find that the number of routes across the 6×6 board is 1683. ■

   Comment: As pointed out by two or three candidates, if the number of diagonal moves is d, where $0 \le d \le 6$, then the number of routes is

$$(12-d)!/d!(6-d)!(6-d)!$$

   This can be summed over d to give 1683.

Comment: I have managed to show that the number of routes from the bottom left to top right squares on an $(n+1) \times (n+1)$ chessboard is slightly more than

$$\binom{2n}{n} \frac{1}{2^{1/4}} \left(\frac{\sqrt{2}+1}{2} \right)^{2n+1} \left\{ 1 + \frac{3\sqrt{2}-4}{32} \cdot \frac{1}{n} + \cdots \right\}.$$

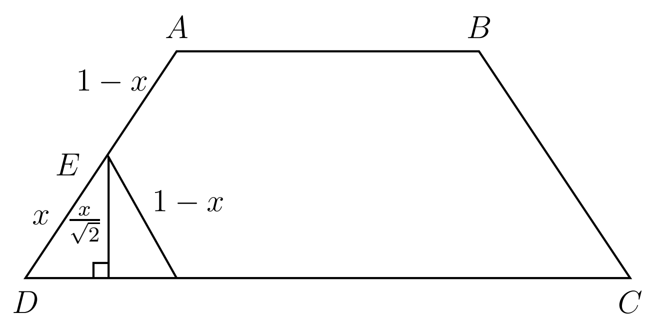
For $n = 1$ this gives $2.9805\ldots$ (the correct answer is 3), and for $n = 5$ the formula gives $1682.6885\ldots$ (the correct answer is 1683).

4. $ABCD$ is a trapezium in which AB is parallel to DC, with

$$AB = BC = DA = 1 \qquad \text{and} \qquad CD = 1 + \sqrt{2}.$$

Let E be a point on AD such that we can fold the trapezium along a line passing through E so that A falls on CD. Find the maximum possible length of DE.

Solution It should first be proved that the trapezium is isosceles, with base angles of 45^o. If we let $DE = x$, then $AE = 1 - x$,



and we obtain the equation

$$1 - x \geq x/\sqrt{2}.$$

This yields

$$x \leq 2 - \sqrt{2}.$$

We can have equality, when the trapezium is folded in such a way that AE becomes perpendicular to CD. ■

5. Eight pieces of string are parallel. We randomly tie four pairs of upper ends and four pairs of lower ends of the strings. What is the probability that all eight pieces of string form just one closed loop?

Solution We can suppose that the tops of the first and second strings are joined, the tops of the third and fourth are joined and so on. The strings form a loop as long as the bottom of the first is tied to the bottom of any string except the second, with probability $6/7$, the bottom of the second is tied to the bottom of any available string except the partner of the string tied to the bottom of the first string, with probability $4/5$, and so on. The probability of a loop is

$$\frac{6}{7} \times \frac{4}{5} \times \frac{2}{3} \times 1 = \frac{16}{35}. \blacksquare$$

6. See Senior (7).

SENIOR DIVISION

1. A pentagon is said to be regular if its five sides all have equal length and if all its angles are equal.

   (i) Show that a cube can be cut by a plane in such a way that the cross–section is a pentagon.

   (ii) Prove that the pentagon is not regular.

   Solution A cube has eight vertices. If a plane intersects the cube so as to separate the eight vertices into sets of one and seven vertices, the cross–section is a triangle; if the plane separates the vertices into sets of two and six, the cross–section is a trapezium or rectangle; if the plane separates the vertices into two sets of four, the cross–section is either a parallelogram or a hexagon with opposite sides parallel. If the plane separates the vertices into sets of three and five, the cross–section is a pentagon with two pairs of parallel sides, so not regular. ■

2. Observe that if $n = 40$, then $2n + 1 = 81$ and $3n + 1 = 121$ are both squares.

   Prove that if n is any positive integer such that both $2n+1$ and $3n+1$ are squares, then n is a multiple of 40.

   Solution Suppose $2n + 1$, $3n + 1$ squares. We shall show that $n \equiv 0 \pmod 5$, $n \equiv 0 \pmod 8$, and so $n \equiv 0 \pmod{40}$.

   First observe that squares are $0,\ 1$ or $4 \pmod 5$, $0,\ 1$ or $4 \pmod 8$.

   If $n \equiv 1 \pmod 5$, $2n + 1 \equiv 3 \pmod 5$, so is not a square.

   If $n \equiv 2 \pmod 5$, $3n + 1 \equiv 2 \pmod 5$, so is not a square.

   If $n \equiv 3 \pmod 5$, $2n + 1 \equiv 2 \pmod 5$, so is not a square.

   If $n \equiv 4 \pmod 5$, $3n + 1 \equiv 3 \pmod 5$, so is not a square. If $n \equiv 1 \pmod 8$, $2n + 1 \equiv 3 \pmod 8$, so is not a square.

   If $n \equiv 2 \pmod 8$, $2n + 1 \equiv 5,\ 3n + 1 \equiv 7 \pmod 8$, so neither is a square.

   If $n \equiv 3 \pmod 8$, $2n + 1 \equiv 7,\ 3n + 1 \equiv 2 \pmod 8$, so neither is a square.

   If $n \equiv 4 \pmod 8$, $3n + 1 \equiv 5 \pmod 8$, so is not a square.

If $n \equiv 5 \pmod 8$, $2n+1 \equiv 3 \pmod 8$, so is not a square.

If $n \equiv 6 \pmod 8$, $2n+1 \equiv 5,\ 3n+1 \equiv 3 \pmod 8$, so neither is a square.

If $n \equiv 7 \pmod 8$, $2n+1 \equiv 7,\ 3n+1 \equiv 6 \pmod 8$, so neither is a square. ■

3. 17 people correspond by mail, each one with the 16 others. Only three different topics are discussed and each pair of correspondents deals with only one of these. Show that there are at least three people who write to one another about the same topic.

   Solution Call one of the people A. A corresponds with 16 others on three topics, so with at least six on one topic, say topic$_1$. Call these people B, C, D, E, F, G. If any two of these correspond on topic$_1$ then A together with these two constitute a set of three who correspond with one another on the same topic. Suppose no two of B, C, D, E, F, G correspond on topic$_1$. Then they correspond on topics 2 and 3. B corresponds with the five others on two topics, so with at least three on one topic, say with C, D, E on topic$_2$. If any two of C, D, E correspond on topic$_2$, then B together with these two constitute a set of three who correspond with one another on the same topic. Otherwise, C, D, E all correspond on topic$_3$ and constitute a set of three who correspond with one another on the same topic. ■

   Comment: The critical numbers $6,\ 17$ are the first of a sequence which depend on the number t of topics. For $t = 2,\ n = 6$, if $t = 3,\ n = 17$, if $t = 4,\ n = 66$ and so on. The numbers are given by

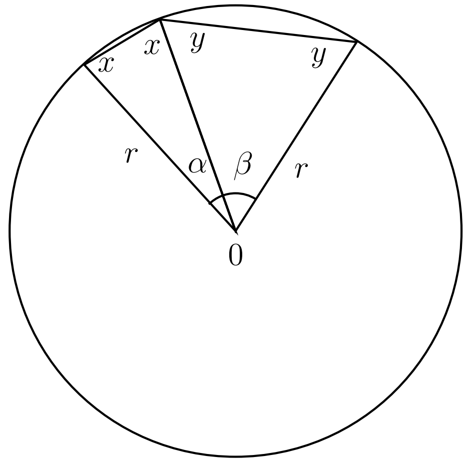
   $$n(t+1) = 1 + n(t) + t(n(t) - 1).$$

4. A convex 12-sided polygon is inscribed in a circle. Six of its sides have length $\sqrt{2}$ and six have length $\sqrt{24}$. What is the radius of the circle?

   Solution Suppose the sides of length $\sqrt{2}$ subtend the angle α at the centre of the circle and the sides of length $\sqrt{24}$ subtend the angle β at the centre of the circle. Then $6\alpha + 6\beta = 360^o$,

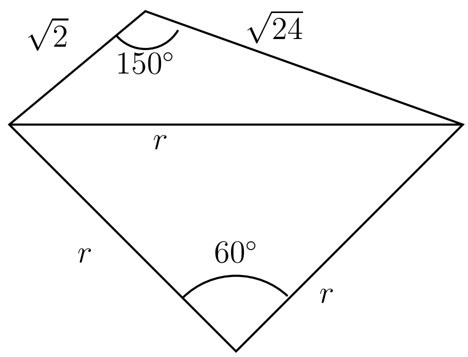
   $$\alpha + \beta = 60^o.$$

   We can draw the following diagram.



Note that $2x+2y+\alpha+\beta = 360^o$, so $x+y=150^o$.

So we have



Thus, from the cosine rule,

$$\begin{aligned} r^2 &= 2+24-2\times\sqrt{2}\times\sqrt{24}\times\cos 150^o \\ &= 2+24-2\times\sqrt{2}\times\sqrt{24}\times -\frac{\sqrt{3}}{2} \\ &= 2+24+12 \\ &= 38. \end{aligned}$$

and $r=\sqrt{38}$. ■

5. The sequence $a_1,\ a_2,\ a_3,\ \cdots$ is defined as follows:

   (i) a_1 is an arbitrary given positive integer.

(ii) For all $n > 1$,

$$a_n = \frac{1}{2}a_{n-1} \text{ if } a_{n-1} \text{ is even}$$
$$a_n = 3a_{n-1} + 1 \text{ if } a_{n-1} \text{ is odd.}$$

Prove that among the members of the sequence there is eventually one which is divisible by 4.

Solution If the binary expansion of a_1 ends in two or more 0's, then a_1 is divisible by 4.

If the binary expansion of a_1 ends in 10 then a_2 is odd, and we can suppose we start with an odd number.

Suppose the binary expansion of a_1 ends in n 1's, preceded by a 0.

Then

$$a_1 \equiv 2^n - 1 \pmod{2^{n+1}}.$$

(For example, if $a_1 = 23$, its binary expansion is 10111, $n = 3$ and $23 \equiv 2^3 - 1 \pmod{2^4}$.)

Then

$$\begin{aligned}
a_2 &= 3a_1 + 1 \equiv 3 \times 2^n - 2 \pmod{2^{n+1}},\\
a_3 &\equiv 3 \times 2^{n-1} - 1 \pmod{2^n},\\
a_4 &= 3a_3 + 1 \equiv 9 \times 2^{n-1} - 2 \pmod{2^n},\\
a_5 &\equiv 9 \times 2^{n-2} - 1 \pmod{2^{n-1}},\\
a_7 &\equiv 3^3 \times 2^{n-3} - 1 \pmod{2^{n-2}},\\
&\vdots\\
a_{2n-3} &\equiv 3^{n-2} \times 2^2 - 1 \pmod{2^3},\\
a_{2n-1} &\equiv 3^{n-1} \times 2 - 1 \pmod{4},\\
a_{2n} &= 3a_{2n-1} + 1 \equiv 3^n \times 2 - 2 \pmod{4},
\end{aligned}$$

or,

$$a_{2n} \equiv 2(3^n - 1) \pmod{4}.$$

Now, $3^n - 1$ is even, so a_{2n} is divisible by 4. ■

6. Find all real–valued functions f defined for x between 0 and 1 which satisfy

$$f(xy) = xf(x) + yf(y).$$

Solution First observe that

$$f(x^2) = xf(x) + xf(x) = 2xf(x).$$

Therefore

$$\begin{aligned} f(x^4) &= x^2 f(x^2) + x^2 f(x^2) \\ &= 2x^2 f(x^2) \\ &= 2x^2 .2x f(x) = 4x^3 f(x). \end{aligned}$$

Also

$$\begin{aligned} f(x^3) &= x^2 f(x^2) + x f(x) \\ &= x^2 .2x f(x) + x f(x) \\ &= (2x^3 + x) f(x) \end{aligned}$$

and

$$\begin{aligned} f(x^4) &= x^3 f(x^3) + x f(x) \\ &= x^3 .(2x^3 + x) f(x) + x f(x) \\ &= (2x^6 + x^4 + x) f(x). \end{aligned}$$

So we have

$$4x^3 f(x) = (2x^6 + x^4 + x) f(x).$$

It follows that $f(x) = 0$ unless $4x^3 = 2x^6 + x^4 + x$. This happens at only one point in $(0, 1)$, $x \approx 0.572$.

However,

$$f(x^5) = x^4 f(x^4) + x f(x) = (4x^7 + x) f(x)$$

and

$$f(x^5) = x^3 f(x^3) + x^2 f(x^2) = (2x^6 + x^4 + 2x^3) f(x)$$

so $f(x) = 0$ unless $4x^7 + x = 2x^6 + x^4 + 2x^3$, and this happens only once, at a different point, $x \approx 0.633$. ■

7. The 64 tennis players, $t_1,\ t_2,\ \cdots\ , t_{64}$ of a certain country are ranked, with t_1 the highest ranked player, t_2 the next, and so on, and t_{64} the lowest ranked. If $j - i > 2$, then t_i always defeats t_j.

   The 64 players take part in a 6-round knockout tournament (that is, after every round the losers take no further part). The draw in each round is decided by a random process. What is the lowest possible ranking of the eventual winner (that is, if t_j wins the tournament, find the largest possible value of j)?

   Prove your assertion.

   Solution The lowest possible ranking of the winner is 12. To see that t_{12} can win, suppose all the even–numbered players are in one half of the draw, the odd–numbered players in the other. In the first round, t_1 and t_2 are knocked out by t_3 and t_4 respectively, in the second round t_3 and t_4 are knocked out by t_5 and t_6

who survived easy matches in the first round, and so on, and after five rounds, all of $t_1, \cdots, t_{10}$ are knocked out. In the final, t_{12} beats t_{11}. To see that no lower–ranked player can win, observe that one of the top ten players will make it to the final unless t_1 and t_2 are knocked out in the first round, t_3 and t_4 are knocked out in the second and so on, and in that case, the lowest ranked player who can win is t_{12} (against t_{10} in the final). ■