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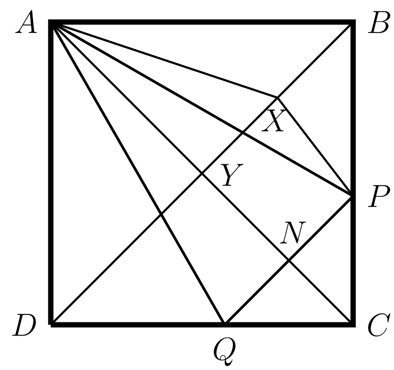
SOLUTIONS TO PROBLEMS 1043–1050

Q.1043 An equilateral triangle APQ is drawn so that P, Q are on the sides BC and DC of a square $ABCD$, with $|AP| = |AQ|$. Show that the perimeter of APQ is less than the perimeter of the triangle ABD (unless P is at B and Q is at D).

ANS.



Suppose the diagonals meet at Y and let $|AB| = a,\ |BY| = d$. Then the perimeter of ABD is $2(a + d)$. If we draw PX parallel to AC, then the triangle BXP is isosceles, and

$$\begin{aligned} a + d &= |AB| + |BY| \\ &= |AB| + |BX| + |XY| \\ &= |AB| + |XP| + |PN| \text{ as } PXYN \text{ is a parallelogram} \\ &\geq |AX| + |XP| + |PN| \\ &\geq |AP| + |PN| \end{aligned}$$

which is half the perimeter of the triangle APQ.

Q.1044 Two missiles are heading towards each other, one at 21,000 km per hour and the other at 39,000 km per hour. If they start 1,319 km apart, how far apart will they be when they collide?

ANS. The two missiles approach each other with combined speed of 60,000 km per hour, or 1000 km per minute. By running the scene backward in time we see that one minute before the collision the missiles would have to be 1000 km apart.

Q.1045 The airforce of a certain island country decide to fly a plane around the world without landing. Since each plane can only hold enough fuel to fly halfway around the world and they can only get fuel from their own island, they decide to send extra planes and transfer fuel in flight (the extra plane or planes returning home each time to refuel). What is the minimum number of planes they can use?

ANS. Three planes are quite sufficient to ensure the flight of one plane around the world. There are many ways this can be done, but the following seems to be the most efficient .

Planes A, B and C take off together. After going $1/8$ of the distance around the world, C transfers $1/4$ tank to A and $1/4$ to B. This leaves C with $1/4$ tank; just enough to get back home.

Planes A and B continue another $1/8$ of the way, then B transfers $1/4$ tank to A. B now has $1/2$ tank left, which is sufficient to get back to the base where it arrives with an empty tank.

Plane A, with a full tank, continues until it runs out of fuel $1/4$ of the way from the base. It is met by C which has been refueled at the base. C transfers $1/4$ tank to A, and both planes head for home.

The two planes run out of fuel $1/8$ of the way from the base, where they are met by refueled plane B. Plane B transfers $1/4$ tank to each of the other two planes. The three planes now have just enough fuel to reach the base with empty tanks.

Q.1046 Smith, Brown and Jones agree to fight a three-way duel. They stand at the vertices of an equilateral triangle and each in turn fires a pistol at either (but not both) of the two. If Smith has a 100% accuracy rate, Brown has an 80% accuracy rate and Jones has a 50% accuracy rate, and each uses the best strategy, who has the best chance of survival?

ANS. Jones, has the best chance to survive. Smith, who never misses, has the second best chance. Because Jones's two opponents will aim at each other when their turns come, Jones's best strategy is to fire into the air until one opponent is dead. He will then get the first shot at the survivor, which gives him a strong advantage.

Smith's survival chances are as follows. There is a chance of $1/2$ that he will get the first shot in his duel with Brown, in which case he kills him. There is a chance of $1/2$ that Brown will shoot first and since Brown is $4/5$ accurate, Smith has a $1/5$ chance of surviving. So Smith's chance of surviving Brown is $1/2$ added to $1/2 \times 1/5 = 3/5$. Jones, who is accurate half the time, now gets a crack at Smith. If he misses, Smith kills him, so Smith has a survival chance of $1/2$ against Jones. Smith's over-all chance of surviving is therefore $3/5 \times 1/2 = 3/10$.

Q.1047 Which of the two number $\sqrt[88]{88!}$ or $\sqrt[99]{99!}$ is the bigger?
(Here $n!$ means the product of the numbers $1, 2, \ldots, n$)?

ANS. Let $x = \sqrt[88]{88!}$ and $y = \sqrt[99]{99!}$. Then (since $88 \times 9 = 99 \times 8 = 792$)

$$\left(\frac{x}{y}\right)^{792} = \frac{(88!)^9}{(99!)^8} = \frac{(88!)^8 \times 88!}{(88!)^8 \times 89 \times 90 \times \cdots \times 99}$$
$$= \frac{88!}{89 \times 90 \times \cdots \times 99} > 1.$$

So $x > y$.

Q.1048 Show that $\cos 72^\circ = (\sqrt{5}-1)/4$.

(**Hint:** $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$).

ANS. $\cos 72^\circ = \sin 18^\circ$ and $5 \times 18^\circ = 90^\circ$. So, if $c = \cos 18^\circ$, then

$$0 = \cos 90^\circ = 16c^5 - 20c^3 + 5c.$$

Since $c \neq 0$,

$$\begin{aligned}16c^4 - 20c^2 + 5 &= 0\\ (2c)^4 - 5 \times (2c)^2 + 5 &= 0\\ (2c)^2 &= \frac{5 \pm \sqrt{5}}{2}\\ c^2 &= \frac{5 \pm \sqrt{5}}{8}.\end{aligned}$$

Since $\cos 18^\circ > \cos 30^\circ \approx 0.8$, we must take the positive square root. So

$$\begin{aligned}\cos^2 18^\circ &= \frac{5+\sqrt{5}}{8}\\ \cos^2 72^\circ = \sin^2 18^\circ &= 1 - \frac{5+\sqrt{5}}{8} = \frac{3-\sqrt{5}}{8}\end{aligned}$$

and $\cos 72^\circ$ is the positive square root of this, i.e. $\cos 72^\circ = (\sqrt{5}-1)/4$.

Q.1049 Suppose that you wish to walk from A to B in steady vertical rain and you want to remain as dry as possible. Should you walk as quickly as possible or as slow as possible or at some intermediate speed? Would it make any difference if you wore a broad brimmed rain hat?

ANS. For simplicity we assume that the person maintains a uniform speed between A and B. Let n denote the number of drops per cubic metre in steady vertical rain. Approximate the geometry of a person by that of a rectangular prism with height h, width w and depth (front to back) d. Let v_D denote the velocity of a rain drop and let v_P denote the velocity of the person. Other relevant parameters are the distance from A to B denoted by s and the angle θ defined by $\tan(\theta) = \frac{|v_D|}{|v_P|}$. The figure shows a two-dimensional cross-section in a plane parallel to the direction of travel. The two shaded parallelograms in this figure show the zones from which rain drops starting at the time the person leaves A will intercept the person on their way to B. The larger parallelogram, with area A_1, is the zone of origin for drops that will hit the person on the front whereas the smaller parallelogram, with area A_2, is the zone of origin for drops that will hit the person on the head.

The total number of drops that will hit the person is thus

$$\begin{aligned}N &= nwA_1 + nwA_2\\ &= nwsh + nwds\tan(\theta)\\ &= nwsh + nwds\frac{|v_D|}{|v_P|}.\end{aligned}$$

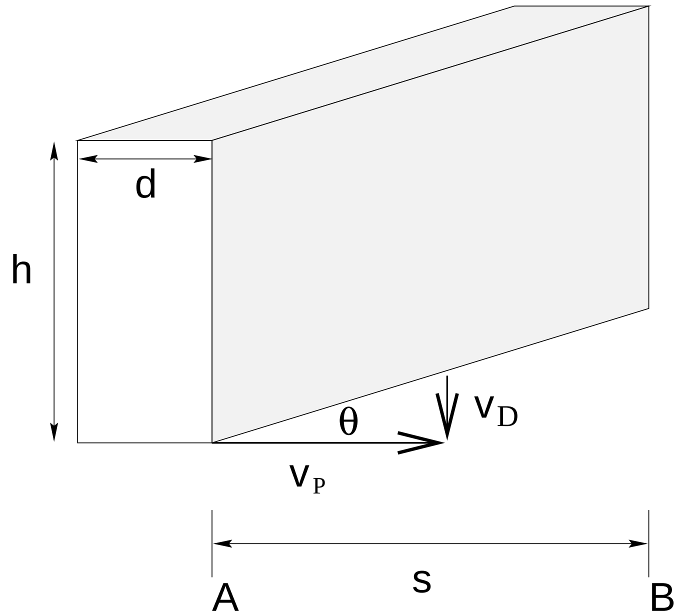


Figure 1: Schematic two-dimensional illustration showing relevant parameters for solving the walking in the rain problem

The first term on the right is the number that will strike the person on the front and the second term is the number that will strike them on the head. The total number is clearly a minimum when v_P is a maximum, i.e., the person should walk (run) as quickly as possible in order to remain as dry as possible. On the other hand if the person is wearing a broad brimmed rain hat then the second term on the right is irrelevant and the total number of drops to hit the person does not depend on their speed.

Moral Buy a good rain hat and go at your leisure.

Q.1050 After her first encounter with the three bears, Goldilocks became their life-long friend and often brought them bags of cookies for a treat. She always gave the largest bag of cookies to father bear, the next largest bag of cookies to mother bear and the smallest bag of cookies to baby bear. The bears always ate all their cookies deliriously never noticing how many cookies each of the others had received.

One day after the bears had eaten their cookies Goldilocks invited the bears to play a game with her. "I will tell you how many cookies I brought in total and then you tell me how many cookies each of the other bears ate". So Goldilocks told them how many cookies she had brought in total and then she began by asking father bear: "How many cookies has each of the other bears eaten?" Father bear replied that he didn't know. Then Goldilocks asked mother bear: "How many cookies has each of the other bears eaten?" Mother bear replied that she didn't know too. Then she asked baby bear the same question but baby bear didn't know either. Then Goldilocks asked father bear again: "How many cookies has each of the other bears eaten?" Father bear replied that he still didn't know. Then Goldilocks asked mother bear again, but before mother bear had a chance to answer baby bear said: "She doesn't know but I do".

Your problem is to work out how many cookies each of the bears ate.

(**Hint:** The bears are clever at arithmetic and Father bear didn't eat more than ten cookies.)

ANS. Father bear had nine cookies, mother bear had six and baby bear had four. The method of solution is via elimination. Let F denote the number of cookies father bear ate, M the number of cookies mother bear ate and B the number of cookies baby bear ate. It is convenient to represent these numbers by a 3-tuple (F, M, B). The total number of cookies $N = F + M + B$.

Since the bears each know the number of cookies in total it is a good strategy to assume an initial total number and then consider all possibilities consistent with the given constraint: $10 \geq F > M > B \geq 1$.

For example, suppose there were ten cookies in total then the set of possibilities is

$$\{(7,2,1), (6,3,1), (5,4,1), (5,3,2)\}.$$

In the first round of questioning father bear does not know how many cookies the other bears had so we can eliminate the possibilities $(7,2,1)$ and $(6,3,1)$ which are unique solutions if father bear had seven or six cookies. Note that mother bear and baby bear can eliminate these possibilities too because they know that father bear did not know how many cookies each bear ate. This leaves the possibilities $\{(5,4,1), (5,3,2)\}$ after father bear says he does not know. We can similarly eliminate $(5,4,1)$ and $(5,3,2)$ after mother bear says that she does not know.

Now consider the solution provided above where $N = 9 + 6 + 4 = 19$. The set of possibilities in this case is

$$\{(10,8,1), (10,7,2), (10,6,3), (10,5,4), (9,8,2), (9,7,3), (9,6,4), (8,7,4), (8,6,5)\}.$$

In the first round of questioning there are no possible unique solutions for father bear and hence father bear clearly does not know. Given that mother bear does not know in the first round we can eliminate the possibility $(10,5,4)$ which yields a unique solution if mother bear had five. Similarly in the first round baby bear does not know so we eliminate $(10,8,1)$ and $(8,6,5)$ which would yield unique solutions if baby bear had one or five cookies. So at the start of the second round of questioning we have the remaining possibilities

$$\{(10,7,2), (10,6,3), (9,8,2), (9,7,3), (9,6,4), (8,7,4)\}.$$

In the second round father bear still does not know so we now eliminate $(8,7,4)$ leaving

$$\{(10,7,2), (10,6,3), (9,8,2), (9,7,3), (9,6,4)\}.$$

Without any further elimination baby bear knows the answer which means baby bear must have had four cookies corresponding to the unique possibility $(9,6,4)$. Note that at this stage there were two remaining possibilities for mother bear, $(10,6,3)$ and $(9,6,4)$, which explains baby bear's response in round two before mother bear had a chance to reply: "She doesn't know but I do".