Parabola Volume 41, Issue 1 (2005)

On the Sum of Powers of Digits of a Number

N. P. Singh[1]

Introduction: In a recent article in *Parabola* (Vol.37, No.3, 2001, 13–16), Anard Kumar proved that the dynamical system defined by summing the squares of the digits of a positive integer repeatedly leads to a fixed point solution $\{1\}$ or a periodic orbit $\{4, 16, 37, 58, 59, 145, 42, 20\}$. The present work identifies fixed points and periodic orbits in the dynamical system defined by summing the rth power of the digits of a positive integer repeatedly.

The central result of this paper is that the dynamical system defined by summing the rth power of the digits of a positive integer repeatedly leads to a finite periodic orbit.

Proof. Let n denote a positive integer consisting of m digits, then

$$m = [\log_{10} n] + 1 < \log_{10} n + 1.$$

The sum of the rth powers of the digits of n is denoted by n_1 and the number of digits in n_1 is denoted by m_1. It now follows that

$$n_1 \leq 9^4.m$$

and

$$\begin{aligned} m_1 &\leq \log_{10}(9^4.m) + 1 \\ m_1 &< r + \log_{10} m + 1. \end{aligned} \tag{1}$$

We call equation (1) the first iteration. Now consider the function

$$f(x) = x - \log_{10} x,$$

which is strictly increasing with x since $f'(x) = 1 - \frac{\log_{10} \mathbf{e}}{x} > 0$. We now have,

$$\begin{aligned} f(r + \log_{10} r + 2) &= r + \log_{10} r + 2 - \log_{10}(r + \log_{10} r + 2) \\ &= r + 2 - \log_{10}\left(1 + \frac{\log_{10} r}{r} + \frac{2}{r}\right) > r + 1 \text{ for } r \leq 1, \end{aligned}$$

$$\text{therefore } x - \log_{10} x > r + 1 \text{ for all } x \geq r + \log_{10} r + 2 \text{ for all } r \geq 1. \tag{2}$$

Case 1: $m \geq r + \log_{10} r + 2$

In this case, $m - \log_{10} m > r + 1$ [using (2)]

$\Rightarrow \quad m > r + 1 + \log_{10} m > m_1.$ [using (1)]

[1]Narendra Singh is a reader from Bihar, India.

Thus every iteration decreases the numbers of digits. Hence after a finite number of iterations, we get a number whose numbers of digits is less than $r + \log_{10} r + 2$.

Case 2: $m < r + \log_{10} r + 2$

In this case using (1),

$$m_1 < r + \log_{10} m + 1 < r + \log_{10}(r + \log_{10} r + 2) + 1$$
$$= r + \log_{10} r + \log_{10}\left(1 + \frac{\log_{10} r}{r} + \frac{2}{r}\right) + 1$$
$$m_1 < r + \log_{10} r + 2.$$

Thus for every starting number n, if we sum the rth powers of the digits successively we get, after a finite number of iterations, a number whose digits are less than $r + \log_{10} r + 2$ in number, and it remains so for all further iterations. As $r + \log_{10} r + 2$ is a fixed number and the iterations can occur as many times as we please, we will eventually culminate in a cycle.

We have calculated cycles for $r = 1$ to 6. The cycles of length 1 and of larger lengths are listed below for $r = 1, 2, 3$ and 4. We have not listed cycles for $r = 5$ and 6, for space considerations. For $r = 5$, we got 7 cycles of length 1 and 9 cycles of greater length. For $r = 6$, we got 2 cycles of length 1 and 5 longer cycles.

The table below shows terminating cycles for the dynamical system defined by summing the rth powers of the digits of a positive integer successively.

Value of r	**Cycles of length 1**	**Cycles of length greater than 1**
1	1,2,3,4,5,6,7,8,9	
2	1	a) 4–16–37–85–89–145–42–20–4 [Anand Kumar*]
3	1,153,370,371,407	a) 1459–919–1459 b) 55–250–133–55 c) 160–217—352–160
4	1,1634,8208,9474	a) 2178–6514–2178 b) 1138–4179–9219–13139–6725–4338–4514–1138

Acknowledgement:

I am very thankful to Dr. D. P. Verma for his guidance.