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Solutions to Problems 1161-1170

Q1161 Find all values of x (real number) satisfying

$$\begin{aligned}&\frac{x-1}{2004}+\frac{x-3}{2002}+\frac{x-5}{2000}+\cdots+\frac{x-2003}{2}\\=&\frac{x-2}{2003}+\frac{x-4}{2001}+\frac{x-6}{1999}+\cdots+\frac{x-2004}{1}\end{aligned}$$

ANS. There are several ways to solve the equation.

1. We can rewrite the equation as
$$\begin{aligned}&x\left(\frac{1}{2004}-\frac{1}{2003}+\frac{1}{2002}-\frac{1}{2001}+\cdots+\frac{1}{2}-\frac{1}{1}\right)\\=&\frac{1}{2004}-\frac{2}{2003}+\frac{3}{2002}-\frac{4}{2001}+\cdots+\frac{2003}{2}-\frac{2004}{1}.\end{aligned}$$
Performing the subtraction of each pair of fractions yields:
$$\begin{aligned}&x\left(-\frac{1}{2004\times 2003}-\frac{1}{2002\times 2001}-\cdots-\frac{1}{2\times 1}\right)\\=&-\frac{2005}{2004\times 2003}-\frac{2005}{2002\times 2001}-\cdots-\frac{2005}{2\times 1},\end{aligned}$$
or equivalently
$$\begin{aligned}&x\left(\frac{1}{2004\times 2003}+\frac{1}{2002\times 2001}+\cdots+\frac{1}{2\times 1}\right)\\=&2005\left(\frac{1}{2004\times 2003}+\frac{1}{2002\times 2001}+\cdots+\frac{1}{2\times 1}\right).\end{aligned}$$
Dividing both sides by $A=\dfrac{1}{2004\times 2003}+\dfrac{1}{2002\times 2001}+\cdots+\dfrac{1}{2\times 1}$, gives $x=2005$.

2. Since each side of the equation has exactly 1002 terms, we can subtract one from each term and obtain
$$\begin{aligned}&\left(\frac{x-1}{2004}-1\right)+\left(\frac{x-3}{2002}-1\right)+\cdots+\left(\frac{x-2003}{2}-1\right)\\=&\left(\frac{x-2}{2003}-1\right)+\left(\frac{x-4}{2001}-1\right)+\cdots+\left(\frac{x-2004}{1}-1\right),\end{aligned}$$

or equivalently

$$\frac{x-2005}{2004}+\frac{x-2005}{2002}+\cdots+\frac{x-2005}{2}$$
$$=\frac{x-2005}{2003}+\frac{x-2005}{2001}+\cdots+\frac{x-2005}{1}.$$

Bringing all terms to one side yields

$$(x-2005)\left[\left(\frac{1}{2003}-\frac{1}{2004}\right)+\left(\frac{1}{2001}-\frac{1}{2002}\right)+\cdots+\left(\frac{1}{1}-\frac{1}{2}\right)\right]=0.$$

$$\text{Let}\quad A=\left(\frac{1}{2003}-\frac{1}{2004}\right)+\left(\frac{1}{2001}-\frac{1}{2002}\right)+\cdots+\left(\frac{1}{1}-\frac{1}{2}\right).$$

Then it is easy to see that A is positive. Hence the equation

$$A(x-2005)=0$$

implies $x-2005=0,\quad$ i.e. $x=2005$.

Q1162 Two women begin to walk at sunrise, one directly from point A to point B, the other directly from point B to point A. They pass exactly at noon. The first reaches point B at 4pm, the second reaches point A at 9pm.

At what time was sunrise that day?

ANS. Let x be the hour the sun rose, and let v_A and v_B be the speeds of the women leaving point A and point B, respectively. Then

$$(12-x)v_A+4v_A \text{ is the distance covered by } A$$

and

$$(12-x)v_B+9v_B \text{ is the distance covered by } B.$$

Since they pass at noon,

$$(12-x)v_A=9v_B$$

and

$$4v_A=(12-x)v_B.$$

Dividing these equations to remove v_A and v_B yields

$$\frac{12-x}{4}=\frac{9}{12-x}$$

or $\quad (12-x)^2=36.\quad$ So $\quad x=6.$

The sun rose at 6am that day.

Q1163 One hundred cows each coloured black or white or brown stand in a field eating 100 bales of hay. Each black cow eats 5 bales, each white cow eats 3 bales, while it takes 3 brown cows to consume 1 bale of hay. Assume that all 100 bales are consumed and that there is at least one cow of each colour. How many cows of each colour are there?

ANS. Let x be the number of black cows, y the number of white cows, and z the number of brown cows. Then

$$x + y + z = 100 \tag{1}$$
$$5x + 3y + \frac{z}{3} = 100. \tag{2}$$

Multiplying (1) by 5 and subtracting (2) from the resulting equation yields

$$2y + \frac{14z}{3} = 400$$

or

$$y = 200 - \frac{7z}{3}\ . \tag{3}$$

Equations (1) and (3) give

$$x = \frac{4z}{3} - 100.$$

So any positive integers x, y, z such that

$$x = \frac{4z}{3} - 100 \quad \text{and} \quad y = 200 - \frac{7z}{3}$$

will be a solution. In order that x and y are integers, 3 must divide z, i.e. $z = 3k$ for $k = 1, 2, 3, \ldots$. Then

$$x = 4k - 100 \quad \text{and} \quad y = 200 - 7k.$$

Condition $x > 0$ implies $k > 25$, and condition $y > 0$ implies $k \leq 28$. So all possible solutions are

$$\begin{array}{llll} k = 26: & x = 4, & y = 18, & z = 78 \\ k = 27: & x = 8, & y = 11, & z = 81 \\ k = 28: & x = 12, & y = 4, & z = 84. \end{array}$$

Q1164 Among all pairs of positive integers (p, q) such that $p + q = 2004$, which pair yields the maximum value $p!q!$ and which pair yields the minimum value $p!q!$? (Recall that for any positive integer n,

$$n! = 1.2.3\ldots(n-1).n.$$
E.g. $4! = 1.2.3.4 = 24.$)

ANS. By symmetry it suffices to consider the following values of (p, q) :

$$(1, 2003),\ (2, 2002),\ (3, 2001), \ldots, (1001,\ 1003),\ (1002,\ 1002).$$

Now note that

$$p!q! = p(p-1)!q!$$

and

$$(p-1)!(q+1)! = (q+1)(p-1)!q!.$$

So

$$p!q! < (p-1)!(q+1)!$$

if and only if

$$p < q + 1$$

or

$$p < 2004 - p + 1$$

or

$$p \leq 1002.$$

Therefore

$$1002!\ 1002! < 1001!\ 1003! < \ldots < 1!\ 2003!$$

Hence $p!q!$ has maximum value when $p = 1,\ q = 2003$ or $p = 2003,\ q = 1$, and has minimum value when $p = q = 1002$.

Q1165 Let n be a natural number and k be the number of distinct primes that divide n. Prove that

$$n \geq 2^k.$$

ANS. If $n = 1$ then there is no prime that divides n. So $k = 0$. Thus $n \geq 2^k$.

Now assume that $n > 1$. Let $p_1, p_2, \ldots, p_k$ be k distinct primes that divide n. Then

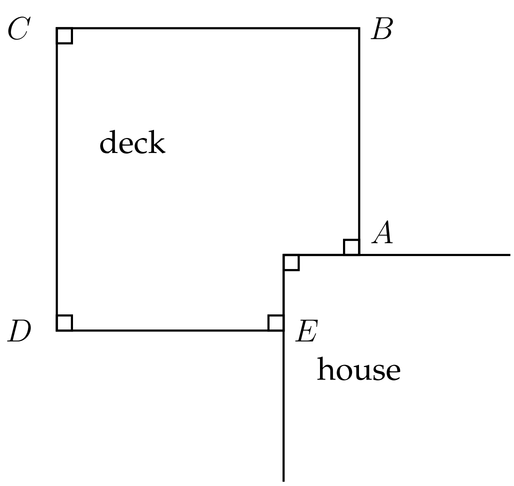
$$n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \ldots p_k^{\alpha_k},$$

where $\alpha_1,\ \alpha_2, \ldots, \alpha_k$ are integers greater than or equal to 1. Also none of the primes $p_1, p_2, \ldots, p_k$ is less than 2. So

$$n \geq 2^{\alpha_1 + \alpha_2 + \cdots + \alpha_k} \geq 2^k.$$

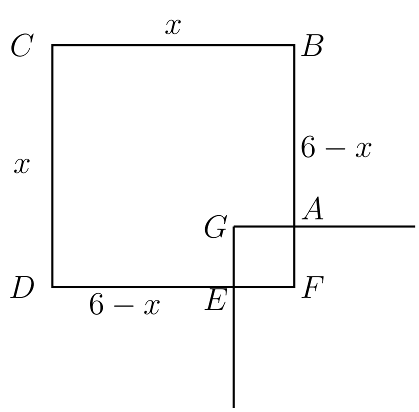
Q1166 Bill wants to build a deck at the corner of his house, as in the figure, where

$AB = DE$ and $BC = CD$. He puts a railing around the outer edges of the deck. Railings are sold in length $6m$ each, and he buys two of them, intending to cut each into two pieces to have the four required railings.



How should he cut to maximise the area of the deck?

ANS. Let $x = BC$. Then $AB = 6 - x$. The area of the deck is then



$$\begin{aligned} S &= \text{area } (BCDF) - \text{ area } (AGEF) \\ &= x^2 - (2x-6)^2 \text{ (since} \\ &\qquad AF = EF = 2x - 6) \\ &= -3(x^2 - 8x + 12) \end{aligned}$$

At this stage if you have learned about parabolas then a sketch of $S = -3(x^2 - 8x + 12)$ reveals that S attains its maximum value when $x = 4$, and the maximum value is 12.
If you haven't yet learned about parabolas you can rewrite S as

$$\begin{aligned} S &= -3[(x-4)^2 - 4] \\ &= -3(x-4)^2 + 12. \end{aligned}$$

Then it is clear that $S \leq 12$ for all x satisfying $0 \leq x \leq 6$, and S attains its maximum value 12 when $x = 4$.

Bill should cut each railing into two pieces of length $4m$ and $2m$, respectively.

Q1167 The lengths of the sides of a triangle form an arithmetic progression with difference $\sqrt{2}$. Assume that the area of the triangle is 12. Prove that it is a right-angled triangle.

ANS. Let the sides of the triangle be a, b, c with $a < b < c$. Then

$$a = b - \sqrt{2} \text{ and } c = b + \sqrt{2},$$

where $b > \sqrt{2}$. If $s = \dfrac{a+b+c}{2}$, then by Heron's formula

$$s(s-a)(s-b)(s-c) = 144.$$

Since

$$s = \frac{b - \sqrt{2} + b + b + \sqrt{2}}{2} = \frac{3b}{2},$$
$$s - a = \frac{b}{2} + \sqrt{2},\; s - b = \frac{b}{2},\; s - c = \frac{b}{2} + \sqrt{2}.$$

We have

$$\frac{3b^2}{4}\left(\frac{b^2}{4} - 2\right) = 144,$$

or

$$b^4 - 8b^2 - 768 = 0.$$

Solving this quadratic equation with unknown b^2 yields

$$b^2 = 32.$$

So the sides of the triangle are

$$\begin{aligned} a &= 4\sqrt{2} - \sqrt{2} = 3\sqrt{2} \\ b &= 4\sqrt{2} \\ c &= 4\sqrt{2} + \sqrt{2} = 5\sqrt{2}. \end{aligned}$$

Since

$$a^2 + b^2 = 50 = c^2,$$

by Pythagoras's Theorem, the triangle is a right-angled triangle.

Q1168 Inequality (2) in Question 1154 (Vol 40, No.1, 2004),

$$\frac{1}{a^3 + b^3 + abc} + \frac{1}{b^3 + c^3 + abc} + \frac{1}{c^3 + a^3 + abc} \le \frac{1}{abc},$$

can be generalised to

$$\frac{1}{a^4 + b^4 + c^4 + abcd} + \frac{1}{b^4 + c^4 + d^4 + abcd} + \\ \frac{1}{c^4 + d^4 + a^4 + abcd} + \frac{1}{d^4 + a^4 + b^4 + abcd} \le \frac{1}{abcd}.$$

Prove this inequality.

ANS. For any positive real numbers x, y, z, there holds

$$\begin{aligned} x^4+y^4+z^4 &= \frac{1}{2}(x^4+y^4)+\frac{1}{2}(y^4+z^4)+\frac{1}{2}(z^4+x^4) \\ &\geq x^2y^2+y^2z^2+z^2x^2 \qquad \text{(Cauchy's inequality)} \\ &= \frac{1}{2}(x^2y^2+y^2z^2)+\frac{1}{2}(y^2z^2+z^2x^2)+\frac{1}{2}(z^2x^2+x^2y^2) \\ &\geq xy^2z+yz^2x+zx^2y \qquad \text{(Cauchy's inequality)} \\ &= xyz(x+y+z). \end{aligned}$$

Applying the above for a, b, c, d yields

$$\begin{aligned} &\frac{1}{a^4+b^4+c^4+abcd}+\frac{1}{b^4+c^4+d^4+abcd}+\frac{1}{c^4+d^4+a^4+abcd} \\ &\quad+\frac{1}{d^4+a^4+b^4+abcd} \leq \frac{1}{abc(a+b+c)+abcd}+\frac{1}{bcd(b+c+d)+abcd} \\ &\quad+\frac{1}{cda(c+d+a)+abcd}+\frac{1}{dab(d+a+b)+abcd} \\ &= \frac{1}{abc(a+b+c+d)}+\frac{1}{bcd(a+b+c+d)} \\ &\quad+\frac{1}{cda(a+b+c+d)}+\frac{1}{dab(a+b+c+d)} \\ &= \frac{d+a+b+c}{abcd(a+b+c+d)} = \frac{1}{abcd}. \end{aligned}$$

Q1169 Given $T_1 = 1$, we define

$$T_{n+1} = 1 + T_1T_2T_3 \ldots T_n \text{ for } n \geq 1.$$

(a) Prove that T_m and T_n are relatively prime integers if $m \neq n$.

(b) Prove that $T_{n+1} = T_n^2 - T_n + 1$ for $n > 1$.

ANS.

(a) Without loss of generality we can assume $m > n$. Let d be a divisor of both T_m and T_n. Then $T_m = kd$ and $T_n = \ell d$ for some $k, \ell \in \mathbb{N}$. Since

$$T_m = 1 + T_1 \,.\, T_2 \ldots T_n \ldots T_{m-1}$$

we have

$$kd = 1 + \ell dr,$$

where $r = T_1 T_2 \dots T_{n-1} T_{n+1} \dots T_{m-1}$. So $(k - \ell r)d = 1$.

Since both d and $k - \ell r$ are natural numbers, we must have $d = 1$. Hence T_m and T_n are relatively prime integers.

(b) For $n > 1$ we have

$$T_{n+1} = 1 + T_1 \dots T_{n-1} \,.\, T_n. \qquad (1)$$

Note that

$$T_n = 1 + T_1 \dots T_{n-1},$$

so

$$T_1 \dots T_{n-1} = T_n - 1. \qquad (2)$$

Substituting (2) into (1) gives

$$\begin{aligned} T_{n+1} &= 1 + (T_n - 1)T_n \\ &= T_n^2 - T_n + 1. \end{aligned}$$

Q1170 Let T_n be defined as in Question 1169.

(a) Prove that

$$\frac{1}{T_1} + \frac{1}{T_2} + \dots + \frac{1}{T_N} = 2 - \frac{1}{T_{N+1} - 1} \quad \text{for all} \quad N \geq 1.$$

(b) (For Senior students) Find the sum of the series

$$\sum_{n=1}^{\infty} \frac{1}{T_n}.$$

ANS.

(a) From (b) of Question 1169 we deduce

$$T_{n+1} - 1 = T_n(T_n - 1) \quad \text{for all} \quad n \geq 2,$$

which implies

$$\frac{1}{T_{n+1} - 1} = \frac{1}{T_n(T_n - 1)} = \frac{1}{T_n - 1} - \frac{1}{T_n}, \quad n \geq 2,$$

or

$$\frac{1}{T_n} = \frac{1}{T_n - 1} - \frac{1}{T_{n+1} - 1}, \quad n \geq 2.$$

So for $N \geq 2$ we have

$$\frac{1}{T_1} + \frac{1}{T_2} + \cdots + \frac{1}{T_N} = \frac{1}{T_1} + \left(\frac{1}{T_2 - 1} - \frac{1}{T_3 - 1}\right) + \left(\frac{1}{T_3 - 1} - \frac{1}{T_4 - 1}\right)$$
$$+ \ldots + \left(\frac{1}{T_{N-1} - 1} - \frac{1}{T_N - 1}\right) + \left(\frac{1}{T_N - 1} - \frac{1}{T_{N+1} - 1}\right).$$

Simplifying the right hand side gives

$$\frac{1}{T_1} + \frac{1}{T_2} + \cdots + \frac{1}{T_N} = \frac{1}{T_1} + \frac{1}{T_2 - 1} - \frac{1}{T_{N+1} - 1}.$$

Since $T_1 = 1$ and $T_2 = 2$ we have

$$\frac{1}{T_1} + \frac{1}{T_2} + \cdots + \frac{1}{T_N} = 2 - \frac{1}{T_{N+1} - 1}.$$

The formula is also true for $N = 1$.

(b) We first note that

$$T_{N+1} = 1 + T_1 \,.\, T_2 \ldots T_N \geq T_2 \ldots T_N \geq 2^{N-1}, \quad N \geq 2.$$

So

$$0 \leq \frac{1}{T_{N+1}} \leq \frac{1}{2^{N-1}}, \quad N \geq 2.$$

Hence $\displaystyle\lim_{N\to\infty} \frac{1}{T_{N+1}} = 0.$

Therefore

$$\sum_{n=1}^{\infty} \frac{1}{T_n} = \lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{T_n} = \lim_{N\to\infty} \left(2 - \frac{1}{T_{N+1} - 1}\right) = 2.$$