Parabola Volume 41, Issue 2 (2005)

Checking Determinants[1]

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When I was a young mathematics student, I often wondered whether there was an easy way of checking determinants. By recently studying the checking of contractants I found there is a fairly easy way to accomplish this.

Suppose you wish to evaluate an nth order determinant D, you expand it to reach your result and of course you are not quite sure whether you are right at this point. Your next move is to construct a check determinant $D1$ as follows. You first add a checkrow which consists of all the column sums of D. If n is odd you now delete row 1 and make your checkrow your last row. Should n be even you also delete row 1 but this time multiply each element of the checkrow by -1. To make the above quite clear let us study the following two illustrations.

Illustration 1. Evaluate and check $D = \begin{vmatrix} 3 & 2 & 1 \\ 2 & 5 & 3 \\ 1 & 4 & 3 \end{vmatrix}$.

We now study the auxiliary system $AS = \begin{vmatrix} 3 & 2 & 1 \\ 2 & 5 & 3 \\ 1 & 4 & 3 \\ 6 & 11 & 7 \end{vmatrix}$,

where we have added the checkrow. As n is odd for D, the checkrow remains unaltered.

Next, we compute D the usual way and obtain that $D = 6$,

finally we study $D1$, our check determinant, as given by $D1 = \begin{vmatrix} 2 & 5 & 3 \\ 1 & 4 & 3 \\ 6 & 11 & 7 \end{vmatrix}$.

Computing $D1$ yields that $D = 6 = D1$, so all is well.

Illustration 2. Evaluate and check $D = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 5 & 7 & 9 & 3 \\ 2 & 1 & 8 & 2 \\ 4 & 2 & 6 & 5 \end{vmatrix}$.

Proceeding the usual way you should find that $D = -4$. As D is an even determi-

[1] If you have not met a determinant before then you should consult the Editor's footnote at the end of this article

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nant we must now reverse all the signs in the checkrow of the check determinant:

$$D1 = \begin{vmatrix} 5 & 7 & 9 & 3 \\ 2 & 1 & 8 & 2 \\ 4 & 2 & 6 & 5 \\ -12 & -11 & -24 & -11 \end{vmatrix}.$$

Computing $D1$ gives $D1 = -4$ also. So now we can be reasonably confident that all should be well.

Reference.

J. Guest, *The Checking of Contractants*, S & M Note 235, Aeronautical Research Laboratories.

Editor's Footnote

Most of you will be familiar with solving systems of linear equations with two or perhaps three unknowns. Solving two simultaneous equations for two unknowns is fairly straightforward but three equations in three unknowns can start to get tedious. But why stop at three? Some problems involve large numbers of equations with large numbers of unknowns. In order to manage these problems it is useful to write the equations in *matrix* form and to use *matrix* reduction methods. As an example, the system of equations

$$\begin{aligned} 2x - 3y + 4z &= 8 \\ 3x - 5y + 2z &= -1 \\ x - 2y + 3z &= 6 \end{aligned}$$

can be written in matrix form as

$$\begin{pmatrix} 2 & -3 & 4 \\ 3 & -5 & 2 \\ 1 & -2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ -1 \\ 6 \end{pmatrix}$$

where the array

$$\begin{pmatrix} 2 & -3 & 4 \\ 3 & -5 & 2 \\ 1 & -2 & 3 \end{pmatrix}$$

is called a matrix. As an exercise you might like to show that this system of equations has the unique solution $x = 1, y = 2, z = 3$.

In general suppose that we have a system of n linear equations to solve for the n unknowns $x_1, x_2, x_3, \ldots x_n$ as follows:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= b_n. \end{aligned}$$

We can re-write the above system in matrix form as

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

Not every system of n linear equations in n unknowns has a unique solution. For example the linear system

$$\begin{aligned} 2x - 3y + 4z &= 8 \\ 3x - 5y + 2z &= -1 \\ x - 2y - 2z &= 5 \end{aligned}$$

does not have a unique solution. Can we determine in advance if the system of equations does have a unique solution? It turns out that we can. We write the linear system in matrix form and then we compute the so called *determinant* of the matrix. Here we will consider the case where the right hand side of the equations is non-zero. If the determinant is equal to zero then the system does not have a unique solution but if the determinant is not equal to zero then there is a unique solution. Moreover, if a unique solution exists then it can be written down using a standard formula (Cramer's Rule) involving determinants. In particular the solution for the unknown x_k can be written as

$$x_k = \det(A_k)/\det(A)$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1(k-1)} & a_{1k} & a_{1(k+1)} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2(k-1)} & a_{2k} & a_{2(k+1)} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{n(k-1)} & a_{nk} & a_{n(k+1)} & \dots & a_{nn} \end{pmatrix}$$

and

$$A_k = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1(k-1)} & b_1 & a_{1(k+1)} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2(k-1)} & b_2 & a_{2(k+1)} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{n(k-1)} & b_n & a_{n(k+1)} & \dots & a_{nn} \end{pmatrix}.$$

and $\det(Z)$ means the determinant of the matrix Z.

We now consider the problem of computing determinants. The determinant of the 2×2 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

is

$$\det(A) \equiv \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \equiv a_{11}a_{22} - a_{12}a_{21}.$$

The determinant of the 3×3 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

is

$$\det(A) \equiv \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \equiv a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

For a general $n \times n$ matrix

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(M_{ij})$$

where M_{ij} is the so-called minor matrix obtained by eliminating the ith row and the jth column from A.

As an example consider the determinant of the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 5 & 7 & 9 & 3 \\ 2 & 1 & 8 & 2 \\ 4 & 2 & 6 & 5 \end{pmatrix}.$$

We begin with the expansion

$$\det(A) = \begin{vmatrix} 7 & 9 & 3 \\ 1 & 8 & 2 \\ 2 & 6 & 5 \end{vmatrix} - \begin{vmatrix} 5 & 9 & 3 \\ 2 & 8 & 2 \\ 4 & 6 & 5 \end{vmatrix} + \begin{vmatrix} 5 & 7 & 3 \\ 2 & 1 & 2 \\ 4 & 2 & 5 \end{vmatrix} - \begin{vmatrix} 5 & 7 & 9 \\ 2 & 1 & 8 \\ 4 & 2 & 6 \end{vmatrix}.$$

We now need to evaluate four determinants and then combine the results as above. Explicitly we have

$$\begin{aligned} \begin{vmatrix} 7 & 9 & 3 \\ 1 & 8 & 2 \\ 2 & 6 & 5 \end{vmatrix} &= 7 \begin{vmatrix} 8 & 2 \\ 6 & 5 \end{vmatrix} - 9 \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} + 3 \begin{vmatrix} 1 & 8 \\ 2 & 6 \end{vmatrix} \\ &= 7\,(40 - 12) - 9\,(5 - 4) + 3\,(6 - 16) \\ &= 157 \end{aligned}$$

and similarly we can evaluate

$$\begin{vmatrix} 5 & 9 & 3 \\ 2 & 8 & 2 \\ 4 & 6 & 5 \end{vmatrix} = 62, \quad \begin{vmatrix} 5 & 7 & 3 \\ 2 & 1 & 2 \\ 4 & 2 & 5 \end{vmatrix} = -9, \quad \begin{vmatrix} 5 & 7 & 9 \\ 2 & 1 & 8 \\ 4 & 2 & 6 \end{vmatrix} = 90$$

so after combining we have $\det(A) = -4$.

Clearly, evaluating determinants in this way is a time consuming task with lots of room for arithmetic errors. How can we be reasonably certain that we have not made an error? The prequel article by J. Guest addresses this issue.