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History of Mathematics: The Invention of Complex Numbers

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In my previous column, I outlined the story of the most recent extension of the number system, so that it expanded to include "infinitesimals," numbers smaller than any of our familiar real numbers, and yet not the same as zero. This time I want to tell of the early days of a similar extension: the introduction of the complex numbers.

When I was in the last years of my own high school studies, I was introduced to complex numbers via quadratic equations. Readers of *Parabola* will most likely be familiar with the story: if the discriminant of a quadratic equation is positive, then the equation has two real roots, while if it is negative, there are two complex roots; the intermediate case of a zero discriminant gives rise to two real roots, but they are equal to one another.

The introduction of the complex numbers means that we can say that every quadratic equation has exactly two roots (although in some circumstances these may happen to be equal. [This is in fact a special case of a more general theorem according to which every n^{th}-degree polynomial equation has exactly n roots, if we allow the possibility of complex roots]).

So when you encountered this nice result, you may well have felt as I did that this was the motivation behind the introduction of complex numbers. However, as I discovered later this is a mistaken belief, and the truth of the matter is much more roundabout.

It was actually the solution of the *cubic* (3^{rd}-degree) equation that stimulated the introduction of "imaginary" and complex numbers. There are many places in which you may read this story.

The website

http://www-history.mcs.st-andrews.ac.uk/Mathematicians/Cardan.html will be accessible to most readers and is reasonably accurate. Another account is given by William Dunham in Chapter 6 of his *Journey through Genius*. Both these retellings of the story give a lot of lurid details that I will omit here. There is a more sober account, emphasizing rather the underlying Mathematics, in JN Crossley's *The Emergence of Number*. Also in a recent (2005) issue of *The Mathematical Intelligencer*, a journal which can be consulted in most university libraries, there is an excellent summary by Kellie Gutman, who provides a lot of interesting details that I leave readers to discover for themselves.

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However, here are the bare bones of the story. The first published account of the solution of a cubic equation is to be found in an algebra text (the very first one) called *Ars Magna* ('The Great Art') by Gerolamo Cardano (Cardan). This appeared in 1545, but much of the work on cubics contained in it was the product of research by Cardano's associates Fior, Fontana ('Tartaglia'), del Ferro and Ferrari. This appropriation of the work of others (although acknowledged) led to some bitter animosities.

But let us leave all that side of things to concentrate on the actual Mathematics. Write the general cubic as:

$$x^3 + ax^2 + bx + c = 0.$$

One original contribution that Cardano *did* make, was the observation that we could always take a to be zero. This leads to important consequences, and so Cardano deserves an honorable place in the story, his other work consisted of retelling the discoveries of others.

Here is why we may take $a = 0$. In the general equation (above), put $x = y + h$, where the value of h remains to be chosen. After some simplification, we find that the equation becomes

$$y^3 + (3h + a)y^2 + (dh^2 + 2ah + b)y + (h^3 + ah^2 + bh) = 0.$$

If we now choose the value $h = -a/3$, this equation reduces to:

$$y^3 + \left(b - \frac{a^2}{3}\right) y + \left(\frac{2a^3}{27} - \frac{ab}{3}\right) = 0,$$

which we can rewrite as

$$y^3 + By + C = 0.$$

This equation has the same form as the original one, except that the coefficient of the squared term has been reduced to zero.

Since Cardano, therefore, all theory of the cubic equation has treated this reduced form, which here will be written

$$x^3 + bx + c = 0.$$

From this point on, there are various accounts, all mathematically equivalent. The one I choose to present here was published in the *International Journal of Mathematical Education in Science and Technology*, February 2003, in a short article by AJB Ward. Ward's idea was to do for the cubic what the standard approach of "completing the square" does for the quadratic.

Recall that for the quadratic

$$x^2 + bx + c = 0$$

we seek to find an equivalent form

$$(x - p)^2 = q^2$$

so that the solutions are $x = p \pm q$. If we compare the two forms, i.e. substitute the hoped-for value of x into the original quadratic, we find:

$$(p^2 + q^2 + bp + c) \pm q(2p + b) = 0$$

and this equation is easily reduced to the two equations

$$p^2 + q^2 + bp + c = 2p + b = 0.$$

We may now solve these equations for p and q, and discover the well-known "formula".

Ward took the cubic

$$x^3 + bx + c = 0$$

and sought to write it as

$$(x - p)^3 = q^3$$

in order to achieve a solution $x = p + \omega q$, where ω is one of the *three* cube roots of 1.

Making the substitution as in the quadratic case, he obtained

$$(p^3 + q^3 + c) + (3\omega pq + b)(p + \omega q) = 0.$$

To reach this particular form, he must have experimented a little. But now he imposed the two conditions

$$p^3 + q^3 + c = 3\omega pq + b = 0.$$

The analogy with the quadratic case is not exact at this point, but it is still rather close.

Ward now had two equations for the unknowns p and q. These can be written

$$p^3 + q^3 = -c \quad \text{and} \quad p^3 q^3 = -\frac{b^3}{27}$$

which brought him to familiar ground. The insight needed here is that p^3 and q^3 are the roots of the quadratic equation

$$y^2 + cy - \frac{b^3}{27} = 0.$$

(This follows because the first of the two equations relates to the *sum* of the roots, the second to their *product*).

So now, together with all the various solvers before and after Cardano, we can take

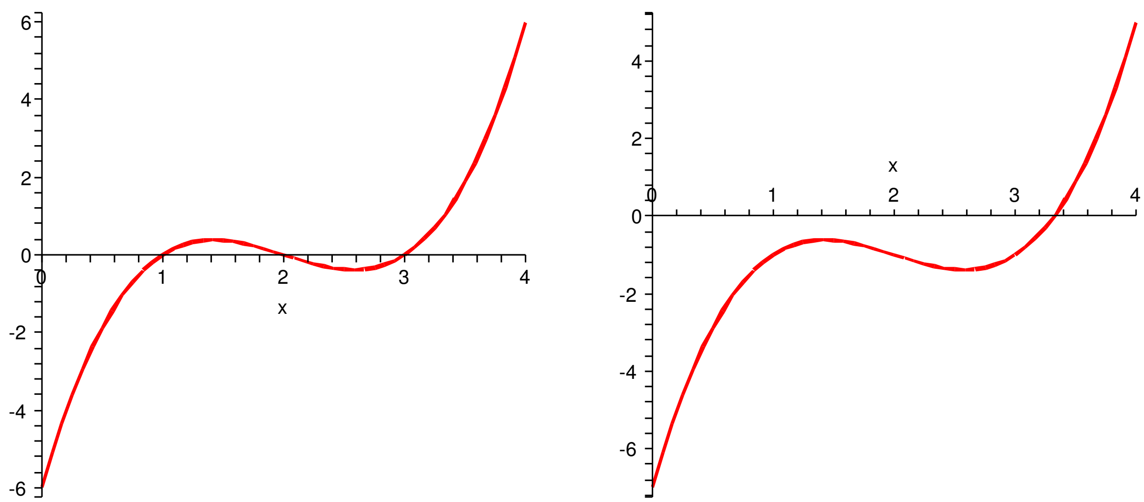
$$p^3 = \frac{1}{2}\left\{-c + \sqrt{c^2 + \frac{4b^3}{27}}\right\}$$

$$q^3 = \frac{1}{2}\left\{-c - \sqrt{c^2 + \frac{4b^2}{27}}\right\}$$

and thus obtain a "formula" for the cubic solution

$$x = \sqrt[3]{\frac{1}{2}\left\{-c+\sqrt{c^2+\frac{4b^3}{27}}\right\}} + \omega\sqrt[3]{\frac{1}{2}\left\{-c-\sqrt{c^2+\frac{4b^3}{27}}\right\}}.$$

Now cubics (again overlooking special cases: two or three equal roots) may be of two types, as shown below. On the left is depicted a cubic with three real roots, on the right, one with one real and two complex roots.



Just as in the case of the quadratic, we seek in the algebra the means to determine which type any particular cubic will belong to. Look back at the formula, and note that there are three possible values that ω may have. 1 and $\frac{-1\pm i\sqrt{3}}{2}$.

So if $c^2 + \frac{4b^3}{27}$ is positive, then two of the roots must be complex.

Let us explore these ideas with the aid of the two examples just illustrated. The one on the left is

$$(x-1)(x-2)(x-3) = 0.$$

whose roots are obvious, while that on the right is

$$(x-1)(x-2)(x-3) = 1,$$

whose roots are not.

Let us look first at the second equation. If we expand and simplify, we reach

$$x^3 - 6x^2 + 11x - 5 = 0.$$

We can put this into the standard (reduced) form by replacing x by $x+2$. This gives

$$x^3 - x + 1 = 0,$$

so that, in the notation used here, $b = -1$ and $c = 1$. Now feed these values into "the formula". The details are messy, but involve only simple arithmetic, and a calculator copes very well. We find that

$$p \approx -0.338, \quad q \approx -0.987.$$

Taking the obvious value for ω, namely 1, now yields $x \approx -1.325$, and if we now recall that we need to add back the 2 that we used to reduce the equation, we end up with a value of approximately 0.675 as the value for the real root of the equation (which may be checked from the diagram). If we also wanted the values of the two complex roots, we could similarly calculate them using the other two values of ω.

Now consider the first of the two examples. The solution is quite obvious, but we want to see how the formula fares. First expand the left-hand side, which gives us

$$x^3 - 6x^2 + 11x - 6 = 0.$$

We now need to put this into reduced form, which means replacing x by $x+2$.

This results in the (still simple) reduced equation

$$x^3 - x = 0.$$

So $b = -1$ and $c = 0$. Now we use the formula to find

$$p = -q = (2i)^{1/3}\frac{b}{3}$$

and so we need to take the cube root of an imaginary number. Our simple example is suddenly starting to look complicated!

However, we may find the cube root involved in this expression. (There are actually three of them, but one will suffice). We can discover that $\left(\frac{\sqrt{3}+i}{2}\right)^3 = i$, and so we have $p = -q = \frac{b\sqrt[3]{2}}{6}(\sqrt{3}+i)$. If we now insert these values into the formula, with $\omega = 1$, we find straightaway, $x = p + q = 0$. This clearly is a root of the reduced equation, and if we use the other roots, we find that $x = \pm 1$ are also solutions. Thus the original equation has roots that are found by adding back the 2, as before, and so we recover the roots 1,2,3.

Such a situation always arises when $c^2 + \frac{4b^3}{27}$ or equivalently $27c^2 + 4b^3$ is negative. It was shown earlier that if this expression was positive then there was only one real root. Similarly, it can be shown that whenever there are three *real* roots, then this expression is negative and vice versa. The expression $27c^2 + 4b^3$ is called the *discriminant* of the cubic, and it plays a role akin to the more familiar discriminant for the quadratic.

The irony is that the supposedly more simple case necessarily involves us in complex numbers. To solve such cubics, we have to use complex numbers there is no escape! This was first convincingly explored by Rafael Bombelli (1526–1572), a somewhat younger contemporary of Cardano, who was employed as an engineer (draining marshes) and worked on Algebra when he had the chance. However, his analysis of such cubics did appear (although probably later than it might have done had he not been busy with other matters) to include a full discussion of the algebra of complex numbers. Indeed he remarked

> "I have found another sort of tied cube root [i.e. $\sqrt[3]{a+\sqrt{b}}$] very different from the others."

So for Bombelli, complex numbers were not a new sort of square root (as they have become for us today), but rather a new sort of *cube root*! He included as an example, the reduced equation $x^3 = 15x + 4$, which he solved by noting that $(2+i)^3 = 2 + 11i$. You can try this example for yourself. (The roots are $4, -2 \pm \sqrt{3}$).

It remains perhaps to say that this case (the irreducible case, as it came to be called) does not yield so readily in all circumstances. The *algebraic* determination of $(a+ib)^{1/3}$ is by no means always possible. This unpleasant truth has been met by means of a subterfuge.

To find such cube roots, express them in polar form and then use de Moivre's Theorem: $(\cos\theta + i\sin\theta)^{1/3} = \cos(\theta/3) + i\sin(\theta/3)$.
Indeed this is what I did earlier in my evaluation of $(2i)^{1/3}$.

We can reflect that this shows a profound link between Algebra and Trigonometry, two (at first sight) very different aspects of Mathematics. On the other hand, it is a confounded nuisance, and it has served to limit the usefulness of the formula. But there's no getting around it, that's just how things are!