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Solutions to Problems 1171–1180

Q1171. The first digit of a 6-digit number is 1. If the 1 is shifted to the other end, the new number is 3 times the original number. Find this number.

ANS: A 6-digit number starting with 1 can be written as

$$100000 + x$$

where x is a 5-digit number. The new number obtained by shifting 1 to the end is

$$10x + 1.$$

By the assumption, there holds

$$10x + 1 = 3(100000 + x),$$

implying $x = 42857$. So the original number is $x = 142857$.

Q1172. Find a number that has 10 divisors such that the product of the divisors is 60466176.

ANS: (submitted by John C. Barton, Victoria).

Since $60466176 = 2^{10}3^{10}$, each divisor can be factorised as $2^{\alpha_1}3^{\alpha_2}$ for some integers $\alpha_1, \alpha_2 \geq 0$. Therefore, the number N to be found has the form $N = 2^{\beta_1}3^{\beta_2}$ for some integers $\beta_1, \beta_2 > 0$. Since α_i can take any value from 0 to $\beta_i, i = 1, 2$, the number of divisors of N is $(\beta_1 + 1)(\beta_2 + 1)$, which is 10 as given. Thus $(\beta_1, \beta_2) = (1, 4)$ or $(\beta_1, \beta_2) = (4, 1)$, i.e., $N = 2 \times 3^4 = 162$ or $N = 2^4 \times 3 = 48$. Neither has divisors with product 60466176. So there does not exist such a number N. (Note: A number having 9 divisors with the same product does exist. It's 36!)

Q1173. (submitted by John C. Barton, Victoria).

In a rectangle, the shorter side is 4 times the difference between the diagonal and the longer side. Find the ratio of the longer side to the shorter side.

ANS: Let a be the length of the shorter side and b be the length of the longer side. Then

$$a = 4(\sqrt{a^2 + b^2} - b),$$

implying

$$a + 4b = 4\sqrt{a^2 + b^2}.$$

By squaring both sides and simplifying we obtain $b/a = 15/8$.

Q1174. Generalise the inequality in Q1168 (Vol 40, No 2, 2004) to the case of n positive numbers a_1, a_2, $\ldots$, a_n, and prove it.

ANS: The generalised inequality is

$$\frac{1}{a_1^n+\cdots+a_{n-1}^2+a_1\cdots a_n}+\frac{1}{a_2^n+\cdots+a_n^2+a_1\cdots a_n}+\cdots \tag{1}$$

$$+\frac{1}{a_n^n+a_1^n+\cdots+a_{n-2}^2+a_1\cdots a_n}\le\frac{1}{a_1\cdots a_n} \tag{2}$$

for $a_1,a_2,\ldots,a_n>0$.

If there holds

$$a_1^n+\cdots+a_{n-1}^n\ge a_1\cdots a_{n-1}(a_1+\cdots+a_{n-1}) \tag{3}$$

then

$$\begin{aligned}\frac{1}{a_1^n+\cdots+a_{n-1}^2+a_1\cdots a_n}&\le\frac{1}{a_1\cdots a_{n-1}(a_1+\cdots+a_{n-1})+a_1\cdots a_n}\\&=\frac{1}{a_1\cdots a_{n-1}(a_1+\cdots+a_n)}\\&=\frac{a_n}{a_1\cdots a_n(a_1+\cdots+a_n)}.\end{aligned}$$

Similarly,

$$\begin{aligned}\frac{1}{a_2^n+\cdots+a_n^2+a_1\cdots a_n}&\le\frac{a_1}{a_1\cdots a_n(a_1+\cdots+a_n)}\\\frac{1}{a_3^n+\cdots+a_n^2+a_1^2+a_1\cdots a_n}&\le\frac{a_2}{a_1\cdots a_n(a_1+\cdots+a_n)}\\&\cdots\cdots\cdots\cdots\cdots\cdots\\\frac{1}{a_n^n+a_1^n+\cdots+a_{n-2}^2+a_1\cdots a_n}&\le\frac{a_{n-1}}{a_1\cdots a_n(a_1+\cdots+a_n)}.\end{aligned}$$

(The missing term in the denominator of the left-hand side appears in the numerator of the right-hand side.) Adding all the relevant inequalities and simplifying the right-hand side of the resulting inequality, we obtain (2).

We now prove (3). For notational convenience, we prove instead

$$a_1^{n+1}+\cdots+a_n^{n+1}\ge a_1\cdots a_n(a_1+\cdots+a_n). \tag{4}$$

Due to Cauchy's inequality

$$a_1\cdots a_n\le\frac{a_1^n+\cdots a_n^n}{n},$$

(4) is true if there holds

$$a_1^{n+1}+\cdots+a_n^{n+1}\ge\frac{1}{n}(a_1^n+\cdots a_n^n)(a_1+\cdots+a_n). \tag{5}$$

By expanding the right-hand side of (5) and rearranging the inequality we see that (5) is equivalent to

$$\begin{aligned}(n-1)(a_1^{n+1}+\cdots+a_n^{n+1})\ge{}&a_1^n(a_2+\cdots+a_n)\\&+a_2^n(a_3+\cdots+a_n+a_1)+\cdots\\&+a_n^n(a_1+\cdots+a_{n-1}).\end{aligned} \tag{6}$$

The left-hand side and right-hand side of (6) can be rewritten as

$$\begin{aligned}
LHS &= (a_1^{n+1} + a_2^{n+1}) + (a_1^{n+1} + a_3^{n+1}) + \cdots + (a_1^{n+1} + a_n^{n+1}) \\
&+ (a_2^{n+1} + a_3^{n+1}) + (a_2^{n+1} + a_4^{n+1}) + \cdots + (a_2^{n+1} + a_n^{n+1}) \\
&+ (a_3^{n+1} + a_4^{n+1}) + (a_3^{n+1} + a_5^{n+1}) + \cdots + (a_3^{n+1} + a_n^{n+1}) \\
&+ \cdots \\
&+ (a_{n-2}^{n+1} + a_{n-1}^{n+1}) + (a_{n-2}^{n+1} + a_n^{n+1}) \\
&+ (a_{n-1}^{n+1} + a_n^{n+1})
\end{aligned}$$

and

$$\begin{aligned}
RHS &= (a_1^n a_2 + a_2^n a_1) + (a_1^n a_3 + a_3^n a_1) + \cdots + (a_1^n a_n + a_n^n a_1) \\
&+ (a_2^n a_3 + a_3^n a_2) + (a_2^n a_4 + a_4^n a_2) + \cdots + (a_2^n a_n + a_n^n a_2) \\
&+ (a_3^n a_4 + a_4^n a_3) + (a_3^n a_5 + a_5^n a_3) + \cdots + (a_3^n a_n + a_n^n a_3) \\
&+ \cdots \\
&+ (a_{n-2}^n a_{n-1} + a_{n-1}^n a_{n-2}) + (a_{n-2}^n a_n + a_n^n a_{n-2}) \\
&+ (a_{n-1}^n a_n + a_n^n a_{n-1}).
\end{aligned}$$

Inequality (6) is true if the sum of each pair in the parentheses of the LHS is greater than or equal to the sum of the corresponding pair in the RHS. So it suffices to prove

$$a^{n+1} + b^{n+1} \geq a^n b + b^n a \quad \forall a, b > 0.$$

This inequality is easily seen to be true if we note that it is equivalent to

$$(a^n - b^n)(a - b) \geq 0 \quad \forall a, b > 0,$$

which is true. Therefore, (4) is proved.

Q1175. (submitted by John C. Barton and Julius Guest, Victoria).

Recall that if m and n are two integers such that $0 \leq n \leq m$, then $\binom{m}{n} = \frac{m!}{n!(m-n)!}$. Find the sum

$$S = \binom{2005}{0} + 2\binom{2005}{1} + 3\binom{2005}{2} + \cdots + 2006\binom{2005}{2005}.$$

ANS: We have by Newton's binomial expansion

$$(1+x)^{2005} = \sum_{k=0}^{2005} \binom{2005}{k} x^k,$$

so that

$$x(1+x)^{2005} = \sum_{k=0}^{2005} \binom{2005}{k} x^{k+1}.$$

By differentiating both sides of the above identity with respect to x we obtain

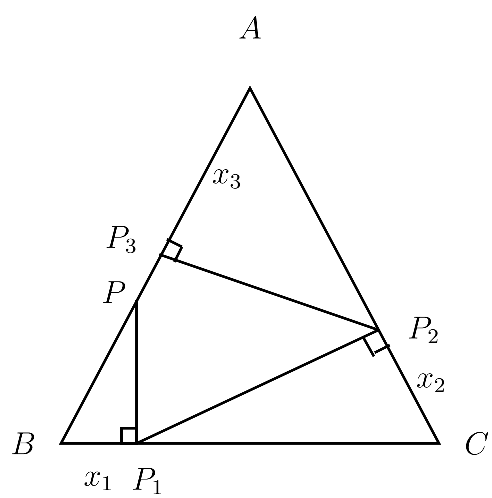
$$(1+x)^{2005} + 2005x(1+x)^{2004} = \sum_{k=0}^{2005} \binom{2005}{k}(k+1)x^k.$$

Letting $x = 1$ we then deduce

$$S = \sum_{k=0}^{2005}(k+1)\binom{2005}{k} = 2^{2005} + 2005 \times 2^{2004} = 2007 \times 2^{2004}.$$

Q1176. Let P be a point on the side AB of an equilateral triangle ABC. Let P_1 be the foot of the perpendicular from P to BC, P_2 be the foot of the perpendicular from P_1 to AC, P_3 be the foot of the perpendicular from P_2 to AB, etc. Show that as n increases indefinitely, the triangle $P_nP_{n+1}P_{n+2}$ is tending to become equilateral.

ANS: Let $BP_1 = x_1$, $CP_2 = x_2$, $AP_3 = x_3$, $BP_4 = x_4$, $CP_5 = x_5$, etc. Let a be the side length of ΔABC.



In ΔP_1CP_2 there holds

$$\frac{x_2}{a - x_1} = \sin 30^o = \frac{1}{2},$$

implying

$$x_2 = -\frac{1}{2}x_1 + \frac{1}{2}a.$$

Similarly, in ΔAP_3P_2 there holds

$$\frac{x_3}{a - x_2} = \sin 30^o = \frac{1}{2},$$

implying

$$x_3 = -\frac{1}{2}x_2 + \frac{1}{2}a = \left(-\frac{1}{2}\right)^2 x_1 + \frac{1}{2}a(1 - \frac{1}{2}).$$

By using induction we can prove that

$$\begin{aligned}
x_{n+1} &= -\frac{1}{2}x_n + \frac{1}{2}a \\
&= \left(-\frac{1}{2}\right)^n x_1 + \frac{1}{2}a\left[1 + \left(-\frac{1}{2}\right) + \left(-\frac{1}{2}\right)^2 + \cdots + \left(-\frac{1}{2}\right)^{n-1}\right] \\
&= \left(-\frac{1}{2}\right)^n x_1 + \frac{1}{2}a\frac{1 - \left(-\frac{1}{2}\right)^n}{1 - \left(-\frac{1}{2}\right)} \\
&= \left(-\frac{1}{2}\right)^n x_1 + \frac{1}{3}a\left[1 - \left(-\frac{1}{2}\right)^n\right].
\end{aligned}$$

As n increases infinitely, $\left(-\frac{1}{2}\right)^n$ approaches 0 and therefore x_n approaches $a/3$. The triangle $P_nP_{n+1}P_{n+2}$ when $x_n = a/3$ is equilateral.

Q1177. Adam and Brian watch the sun setting over the ocean on a calm day. Adam is 10m above the sea level and Brian is on a cliff top 30m above Adam. How long after Adam does Brian observe the instant of sunset. (Take the circumference of the earth to be 40,000km.)

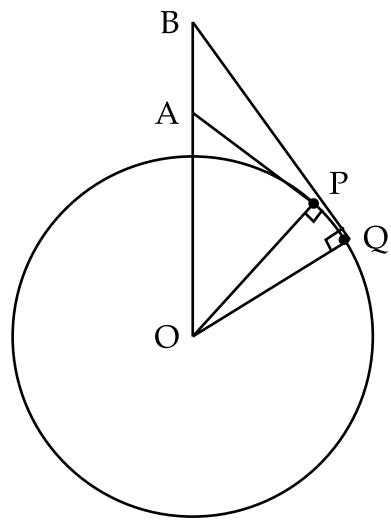
ANS: Let A and B be Adam's position and Brian's position, respectively. Adam spots the sunset when to his eyes the sun is at P, while Brian spots the sunset at Q. The time difference is the time the earth spins through the arc PQ. Since the distances from their positions to the sea level are much smaller than the radius R of the earth, the length of this arc can be approximated by

$$PQ \approx BQ - AP.$$

We have

$$\begin{aligned}
R &= \frac{40000}{2\pi} \approx 6,366\text{km} \\
AP &= \sqrt{OA^2 - OP^2} = \sqrt{\left(R + \frac{1}{100}\right)^2 - R^2} \approx \sqrt{\frac{R}{50}} \approx 11.28\text{km} \\
BQ &= \sqrt{OB^2 - OQ^2} = \sqrt{\left(R + \frac{4}{100}\right)^2 - R^2} \approx \sqrt{\frac{2R}{25}} \approx 22.57\text{km}.
\end{aligned}$$

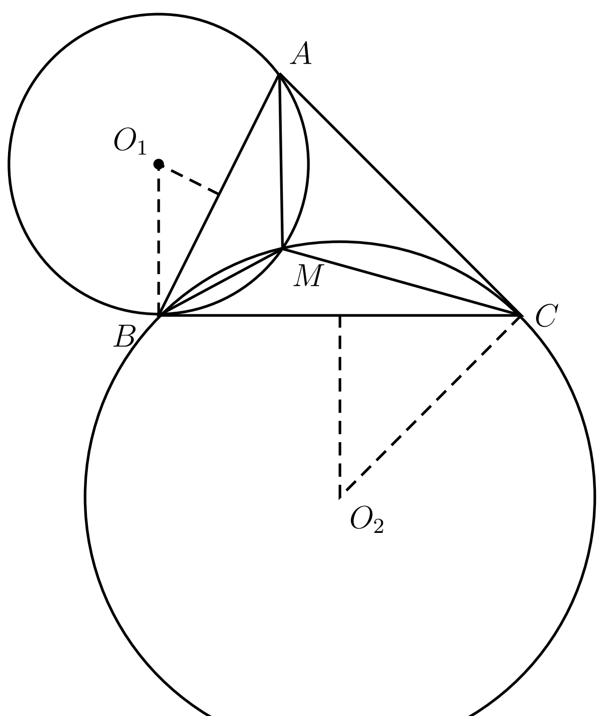
Thus $PQ \approx 11.28$km. The earth spins through 40,000km in one day, i.e., $24 \times 60 \times 60$ seconds. Therefore, it spins through 11.28km in approximately 24.37 seconds. Brian observes the instant of sunset 24.37 seconds after Adam.



Q1178. Find a point M in a triangle ABC satisfying $\angle MAB = \angle MBC = \angle MCA$.

ANS:

1. **Observation:** Suppose that such a point M exists.



If (C_1) is the circle circumscribing ΔMAB, then, since $\angle MAB = \angle MBC$, BC is tangent to (C_1), i.e., $BC \perp O_1B$, where O_1 is the centre of (C_1). Similarly, if (C_2) is the circle circumscribing ΔMBC, centred at O_2, then $AC \perp O_2C$.

2. **To draw** M:

   - Draw O_1 as the intersecting point of the line perpendicular to BC at B and the bisector of AB.
   - Draw O_2 as the intersecting point of the line perpendicular to AC at C and the bisector of AC.
   - Draw the circle centred at O_1 and of radius O_1B.
   - Draw the circle centred at O_2 and of radius O_2B.

   These two circles must cut at another point M (why?).

   This point M lies inside ΔABC (why?) and satisfies

   $\angle MAB = \angle MBC = \angle MCA$.

Q1179. Prove that $(a+b)^2 \geq a^2(1-\epsilon) - b^2(1+1/\epsilon)$ for any $a, b \in \mathbb{R}$ and $\epsilon \in (0,1)$.

ANS: We have

$$2ab = 2(a\sqrt{\epsilon})(b/\sqrt{\epsilon}) \geq -a^2\epsilon - b^2/\epsilon,$$

implying

$$(a+b)^2 = a^2 + b^2 + 2ab \geq a^2(1-\epsilon) + b^2(1-1/\epsilon) \geq a^2(1-\epsilon) - b^2(1+1/\epsilon).$$

Q1180. The function defined by

$$\zeta(n,m) = \sum_{k=1}^{m} \frac{1}{k^n} = \frac{1}{1^n} + \frac{1}{2^n} + \cdots \frac{1}{m^n}$$

where n and m are integers, is an example of a special function in mathematics known as the incomplete Riemann function. Show using elementary operations that

$$\sum_{k=2}^{m} \zeta(n, k-1) = m\zeta(n,m) - \zeta(n-1,m).$$

ANS: First consider the second term on the RHS

$$
\begin{aligned}
\zeta(n-1,m) &= \sum_{k=1}^{m} \frac{1}{k^{n-1}} \\
&= \sum_{k=1}^{m} \frac{1}{k^{n}} \cdot k \\
&= \sum_{k=1}^{m} \frac{1}{k^{n}} \left(m-(m-k)\right) \\
&= m\sum_{k=1}^{m} \frac{1}{k^{n}} - \sum_{k=1}^{m}(m-k)\frac{1}{k^{n}} \\
&= m\zeta(n,m) - \sum_{k=1}^{m}(m-k)\frac{1}{k^{n}}.
\end{aligned}
$$

We can now rearrange the above to write

$$
\begin{aligned}
\sum_{k=1}^{m}(m-k)\frac{1}{k^{n}} &= m\zeta(n,m) - \zeta(n-1,m) \\
\mathbb{R}ightarrow \sum_{k=1}^{m-1}(m-k)\frac{1}{k^{n}} &= m\zeta(n,m) - \zeta(n-1,m). \qquad (1)
\end{aligned}
$$

It is convenient to write the LHS of the above equation as

$$
\begin{aligned}
s(n,m) &= \sum_{k=1}^{m-1}(m-k)\frac{1}{k^{n}} \qquad (2) \\
&= \sum_{k=1}^{m-1}(m-k-1+1)\frac{1}{k^{n}} \\
&= \sum_{k=1}^{m-1}\frac{1}{k^{n}} + \sum_{k=1}^{m-1}(m-k-1)\frac{1}{k^{n}}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{k=1}^{m-1}\frac{1}{k^{n}} + \sum_{k=1}^{m-2}(m-k-1)\frac{1}{k^{n}} \\
&= \zeta(n,m-1) + s(n,m-1).
\end{aligned}
$$

Hence

$$
s(n,m) - s(n,m-1) = \zeta(n,m-1) \qquad m \geq 3
$$

We can now sum over the second index in the above equation to write

$$
\sum_{k=3}^{m} s(n,k) - s(n,k-1) = \sum_{k=3}^{m} \zeta(n,k-1).
$$

But on the LHS of the above equation alternate forms cancel so that all that remains is

$$s(n,m) - s(n,2) = \sum_{k=3}^{m} \zeta(n,k-1)$$
$$\mathbb{R}ightarrow\ s(n,m) = s(n,2) + \sum_{k=3}^{m} \zeta(n,k-1).$$

Finally we note that

$$s(n,2) = \zeta(n,1) = 1.$$

Hence we can write

$$s(n,m) = \sum_{k=2}^{m} \zeta(n,k-1). \tag{3}$$

The results in Equations (1), (2), (3) now combine to give the required result.

This problem and its solution was submitted by N.P. Sing, Bihar, India.