Parabola Volume 41, Issue 3 (2005)

Unsolved Problems[1]

George Szekeres

In the October 1964 issue of *Parabola*, the article on the Four Colour Problem called your attention to the existence of numerous unsolved mathematical problems which can be stated in quite simple non-technical terms. From time to time we wish to write about such "elementary" unsolved problems.

In the same issue, one of our readers, G. Owerchuk, asked what is the largest number n_r of points which may be completely connected with coloured line segments using r different colours in such a way that no one colour triangle results.

For the special case $r = 2$ the problem was set in a previous issue of *Parabola* (Vol. 1, No. 1 Problem 07) and readers were asked to prove $n_2 = 5$. Apart from a few small values of r $(n_1 = 2,\ n_2 = 5,\ n_3 = 16)$ the general solution of the problem is unknown. It can be shown by the same method as used in the solution of 07 that

$$n_r \leq r\, n_{r-1} + 1 \tag{1}$$

For let $c_1, c_2, \ldots, c_r$ be the r given colours, p_0 one of the given points, and s_1 the set of points connected to p_0 by a line segment of colour c_1. No two points $p_1 p_2$ of the set s_1 can be connected by colour c_1, since otherwise $p_0 p_1 p_2$ would be a one colour triangle. Hence the segments connecting points of s_1 can have only $r - 1$ colours, namely $c_2, c_3, \ldots, c_r$. Since they are not supposed to form a triangle of the same colour, s_1 contains at most n_{r-1} points. The same is true of s_2, the set of points connected to p_0 with a segment of colour c_2, etc. Thus the sets $s_1, s_2, \ldots, s_r$, contain altogether at most $r\, n_{r-1}$ points. Since these points together with p_0 exhaust all points of the diagram, we obtain the required inequality.

From the inequality it follows by mathematical induction that

$$n_r \leq r! \left(1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{r!}\right). \tag{2}$$

In fact, for $r = 1$, the inequality becomes $n_1 \leq 1! \left(1 + \frac{1}{1!}\right) = 2$, which is true since obviously $n_1 = 2$. Now let $r > 1$ and suppose that we have already proved that

$$n_{r-1} \leq (r-1)! \left(1 + \frac{1}{1!} + \cdots + \frac{1}{(r-1)!}\right). \tag{3}$$

[1] This article is reprinted from *Parabola* Vol. 2, No. 2, 1965.

Then the inequalities (1) and (3) give

$$\begin{aligned} n_r &\le r\, n_{r-1} + 1 \\ &\le r(r-1)!\left(1 + \frac{1}{1!} + \cdots + \frac{1}{(r-1)!}\right) + 1 \\ &= r!\left(1 + \frac{1}{1!} + \cdots + \frac{1}{(r-1)!} + \frac{1}{r!}\right), \end{aligned}$$

which proves (2).

For $r = 2$, we have $n_2 \le 2!\left(1 + \frac{1}{1!} + \frac{1}{2!}\right) = 5$ and for $r = 3$

$$n_3 \le 3!\left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!}\right) = 16.$$

In both cases there is equality, $n_2 = 5,\ n_3 = 16$. To verify these values it is sufficient to produce a configuration of the required kind with 5 points and 2 colours, or 16 points and 3 colours. An example with 5 points and 2 colours was reproduced in *Parabola* Vol. 1 No.2. The following is a configuration with 16 points and 3 colours, due to Mr. Cox.

We designate the 16 points by

$$\begin{array}{cccccc} p_0 & q_1 & q_2 & q_3 & q_4 & q_5 \\ & r_1 & r_2 & r_3 & r_4 & r_5 \\ & s_1 & s_2 & s_3 & s_4 & s_5 \end{array}$$

The following segments are given the colour c_1,

$$\begin{array}{ccccc} p_0\ q_1, & p_0\ q_2, & p_0\ q_3, & p_0\ q_4, & p_0\ q_5, \\ s_1\ s_2, & s_2\ s_3, & s_3\ s_4, & s_4\ s_5, & s_5\ s_1, \\ r_1\ r_3, & r_3\ r_5, & r_5\ r_2, & r_2\ r_4, & r_4\ r_1, \\ r_1\ s_1, & r_2\ s_2, & r_3\ s_3, & r_4\ s_4, & r_5\ s_5, \\ q_1\ r_3, & q_2\ r_4, & q_3\ r_5, & q_4\ r_1, & q_5\ r_2, \\ q_1\ r_4, & q_2\ r_5, & q_3\ r_1, & q_4\ r_2, & q_5\ r_3, \\ s_1\ q_4, & s_2\ q_5, & s_3\ q_1, & s_4\ q_2, & s_5\ q_3, \\ s_1\ q_5, & s_2\ q_1, & s_3\ q_2, & s_4\ q_3, & s_5\ q_4. \end{array}$$

Those given the colour c_2 are obtained by replacing q_i by $r_i,\ r_i$ by s_i and s_i by q_i in each entry of the previous scheme, and those given the colour c_3 are obtained by a similar cyclic replacement of q_i by $s_i,\ r_i$ by q_i, and s_i, by r_i.

Because of the perfectly symmetrical and cyclic nature of the construction it is quite sufficient to verify that no c_1 triangle exists in which one of the segments is $p_0\ q_1$, or $s_1\ s_2$, or $r_1\ r_3$, or $r_1\ s_1$, or $q_1\ r_3$, or $q_1\ r_4$, or $s_1\ q_4$, or $s_1\ q_5$. This can be done quite easily by inspection of the diagram.

For $r > 3$ nothing definite is known about the problem. It may be conjectured that for every r

$$n_r = r!\left(1 + \frac{1}{1!} + \cdots + \frac{1}{r!}\right),$$

that is, there exists a configuration with that many points when r colours are used. For $r = 4$, there is a known configuration with 41 points (published in 1955 by Greenwood and Gleason in the *Canadian Journal of Mathematics*), but whether this can be bettered to the theoretically best possible value, 65, is not known.

In the next issue we shall deal with another famous unsolved problem on configurations.

Commentary by Catherine Greenhill

The open question described in this issue falls in the area of Ramsey Theory. The definition of a complete graph K_n on n vertices is given in the article on Ramsey Numbers on p.11. The question is, what is the largest value of n such that the edges of K_n can be coloured with r colours without forming a monochromatic triangle (coloured with just one colour)? This largest value of n is denoted by n_r.

When $r = 2$ this is very closely related to the Ramsey number $R(3)$ described in the article on Ramsey Numbers. In fact $R(3) = n_2 + 1$ (convince yourself that this is true). So we know that $n_2 = 5$. George's description of the problem included a proof of this recurrence:

$$n_r \leq r n_{r-1} + 1.$$

From this he derived the upper bound

$$n_r \leq r!\,(1 + 1/1! + 1/2! + \cdots + 1/r!)$$

and conjectured that this bound might be correct for $r \geq 2$. One known result was mentioned: $n_3 = 16$ as proved by Greenwood and Gleason in 1955.

This is still an extremely difficult problem and there has been very little progress. No other exact values of n_r are known. We know that

$$50 \leq n_4 \leq 61.$$

The lower bound was proved by Fan Chung in 1973 and the upper bound was proved by Fettes, Kramer and Radziszowski in 2004. In particular, this shows that George's conjecture was incorrect since it predicted that $n_4 = 65$, which is too big.

This problem has been generalised in many ways. In particular, let N be the largest value of n so that the edges of K_n can be coloured red, blue and purple such that there is no red triangle, no blue triangle and no purple copy of K_4. It is known that

$$30 \leq N \leq 31.$$

The lower bound was proved by Kalbfleisch in 1966 and the upper bound was proved by Piwakowski and Radziszowski in 2001. Since the upper and lower bound are so close, perhaps this particular problem will soon be solved.