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Ramsey numbers

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How many people must attend a party before you are sure that you can find either three people who all know each other, or three people who do not know each other? This is a question in an area called Ramsey Theory.

We can rephrase this question in terms of *graphs*. The *complete graph* of size n, denoted by K_n, consists of a set of n objects, called vertices, such that all $n(n-1)/2 = \binom{n}{2}$ unordered pairs of vertices form an *edge*. We draw a graph with black dots for vertices and a line between two vertices for an edge. Figure 1 shows the complete graphs K_3, K_4 and K_5 on 3, 4 and 5 vertices, respectively. Now we can model the party using the

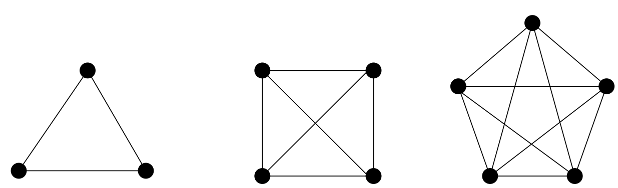

Figure 1: The complete graphs K_3, K_4 and K_5

complete graph K_n, where n is the number of party-goers. Colour the edge from x to y red if x and y knew each other before the party, and blue otherwise. A subgraph is called red if all its edges are red, and similarly for blue subgraphs. Our question now asks whether the graph contains either a red triangle or a blue triangle.

More generally, for integers $s, t \geq 2$ let $R(s,t)$ be the least positive integer n such that any red-blue colouring of the edges of K_n contains either a red K_s or a blue K_t. Set $R(s,t) = \infty$ if no such minimum n exists. Write $R(s) = R(s,s)$ for the diagonal case. The numbers $R(s,t)$ are called *Ramsey numbers*.

What can we say about these Ramsey numbers? It is not difficult to see that $R(s,t) = R(t,s)$. In 1930, Frank Ramsey proved that the diagonal Ramsey numbers $R(s)$ are finite for all $s \geq 2$.

Exercise:

(i) Prove that $R(3) = 6$. This answers the question stated at the start of the article.

(ii) Prove that $R(s,2) = R(2,s) = s$.

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George Szekeres, together with Paul Erdös, proved a very nice recurrence for Ramsey numbers, which we will now describe.

Theorem (Erdös & Szekeres 1935). If $s > 2$, $t > 2$ then

$$R(s,t) \leq R(s-1,t) + R(s,t-1). \tag{1}$$

Proof. Note that $R(r,2) = R(2,r) = r$ is finite, for all $r \geq 2$. So we may assume, by induction on ℓ that $R(r,\ell)$ is finite, for some $\ell \geq 2$. In particular, we can assume that that $R(s-1,t)$ and $R(s,t-1)$ are finite. Once we establish (1) this will prove that $R(s,t)$ is finite, and the induction can continue.

Now set $n = R(s-1,t) + R(s,t-1)$. Consider any red-blue colouring of the edges of the complete graph K_n. We want to show that in this colouring, there is either a red K_s or a blue K_t. Let x be a vertex of K_n. Since x is contained in $n-1$ edges, and $n-1 = R(s-1,t) + R(s,t-1) - 1$, either (a) there are at least $n_1 = R(s-1,t)$ red edges incident with x, or (b) there are at least $n_2 = R(s,t-1)$ blue edges incident with x. By symmetry we may assume that the first case holds.

Next, consider a subgraph K_{n_1} spanned by the n_1 vertices which are joined to x by red edges. If K_{n_1} contains a blue K_t, then we are done. So, suppose that K_{n_1} does not contain a blue K_t. Then, by the definition of $R(s-1,t)$, the graph K_{n-1} must contain a red K_{s-1}. This subgraph, together with x, forms a red K_s. This completes the proof of (1). ∎

Exercise: Use this recurrence to prove that

$$R(s,t) \leq \binom{s+t-2}{s-1}$$

for all $s,t \geq 2$.

This gives an upper bound for the Ramsey numbers $R(s,t)$. But very few of the Ramsey numbers are known precisely. As mentioned above, $R(3) = 6$. It can be shown that

$$R(3,4) = 9, \quad R(3,5) = 14, \quad R(3,6) = 18, \quad R(3,7) = 23,$$

and

$$R(3,8) = 28, \quad R(3,9) = 36, \quad R(4) = 18, \quad R(4,5) = 25.$$

All other Ramsey numbers are beyond the limit of current computing power to determine.

Paul Erdös had a story which illustrated the difficulty of calculating the Ramsey numbers. Here is a paraphrased version: suppose that evil aliens land on the earth and say that they are going to come back in five years and blow it up, *unless* humankind can tell them the value of $R(5)$ when they come back. Then all the mathematicians and computer scientists of the world should get together, and using all the computers in the world, we would probably be able to compute $R(5)$ and save the earth. But what if the aliens had instead said that they would blow up the earth unless we could calculate $R(6)$ in five years? In that case, the best strategy that humankind could follow would be to divert everyone's energy and resources into *weapons research* for the next five years, because finding $R(6)$ is just too hard.

The *diagonal Ramsey numbers* $R(s) = R(s, s)$ are of the most interest. When $s = t$ the upper bound is of the form

$$R(s) \leq \binom{2s-2}{s-1}.$$

Using something called Stirling's formula, this bound implies that

$$R(s) \leq \frac{2^{2s-2}}{\sqrt{s}}.$$

So this gives an upper bound of about $4^s/\sqrt{s}$ for the rate of growth of the diagonal Ramsey numbers. In 1988 Thomassen proved that $R(s) \leq 4^s/s$, a slight improvement. This is the best known upper bound for the diagonal Ramsey numbers. But the best known lower bounds are much much smaller: it is known that $R(s) \geq s2^{s/2}$. The enormous gap between the lower bound and upper bound intrigues many mathematicians, but it seems very difficult to bring these bounds any closer together.