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The Cycle Double Cover Conjecture

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George Szekeres made many contributions to various areas of mathematics. In combinatorics, one of his most important contributions was to ask a question which we still don't know how to answer. His question is the *Cycle Double Cover conjecture*, one of the most famous conjectures in graph theory. In fact, the Cycle Double Cover conjecture was independently posed by George Szekeres in 1973 and by another mathematician, Paul Seymour, in 1979. (This is not uncommon in mathematics: the same idea or question can arise at around the same time, on opposite sides of the world.) George Szekeres was a Professor at the University of New South Wales in Sydney when he published the conjecture.

One reason that the cycle double cover is so interesting is that it has connections with many other areas of graph theory, including the famous Four Colour Theorem.

Background

First, what is a graph? A graph G is a set V of objects, called *vertices*, together with another set E whose elements are unordered pairs $\{u, v\}$ of distinct elements of V. Each such unordered pair is called an *edge*. We draw a graph on the page by representing each vertex by a black dot, and each edge by a line between the two endvertices. An example of a graph H is given in Figure 1.

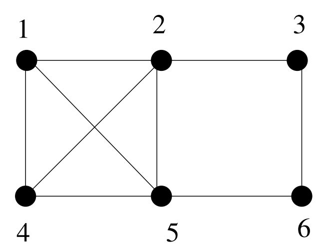

Figure 1: An example of a graph, H

This graph H has 6 vertices and 9 edges. The vertices are labelled by the letters a, b, c, d, e, f. We will use the graph H to introduce a few more graph theoretical concepts which we will need.

Two vertices are said to be *adjacent*, or *neighbours*, if they form an edge. In our graph H, vertex a has neighbours b, d and e. The number of neighbours of a vertex is

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the *degree* of that vertex. In H, vertices c and f have degree 2, vertices a and d have degree 3 and vertices b and e have degree 4. A graph is called *regular* if every vertex has the same degree. So our graph H is not regular. If a graph is regular and every vertex has degree d then we say that the graph is d-regular. A 3-regular graph is also called a *cubic* graph.

A graph is called *connected* if you can walk from one vertex to any other vertex along edges of the graph. For example, in H there is a path from a to f using the edges $\{a, e\}$, $\{e, f\}$. (There are other paths from a to f, but this is the shortest one.) A *cycle* is a walk along at least 3 edges in the graph which starts and ends at the same vertex but which does not revisit any other vertex on the path. In the graph H, the edges

$$\{a, b\}, \{b, e\}, \{e, d\}, \{d, a\}$$

form a cycle, which we also write as $abeda$. It doesn't really matter which vertex you choose as your starting point and which direction you take: for example, $adeba$, $bedab$ and $badeb$ all describe the same cycle as $abeda$. However, $abdeba$ is not a cycle since it visits the vertex b twice.

The conjecture

Now we can start to define the Cycle Double Cover conjecture. A *cycle double cover* of a graph G is a list of cycles in G such that every edge in G is contained in exactly two cycles in the list. We do not insist that all cycles in the list have to be different.

For example, the graph H shown in Figure 1 has a cycle double cover given by the cycles

$$abeda, \qquad bcefb, \qquad abdea, \qquad adbcfea.$$

Check for yourself that this really is a cycle double cover of H.

Exercise: Find a different cycle double cover of H. (Hint: there is a cycle double cover of H with 5 cycles.)

We want to know when a given graph has a cycle double cover. Put another way, what kind of property can guarantee that a given graph G does *not* have a cycle double cover? Well, a cycle double cover is meant to 'cover' every edge of the graph exactly twice, meaning that each edge of the graph must belong to two cycles in the list. But suppose that the graph G has an edge which does not belong to *any* cycle. Then it is impossible to find a cycle double cover of G.

An edge which does not belong to any cycle is called a *cut edge* or *bridge* of G. Deletion of a cut edge from the edge set E (without deleting any vertices) disconnects the graph into two pieces. Figure 2 shows a graph with a cut edge, and illustrates that deleting the cut edge disconnects the graph into two pieces. (This time we do not show the labels of the vertices.)

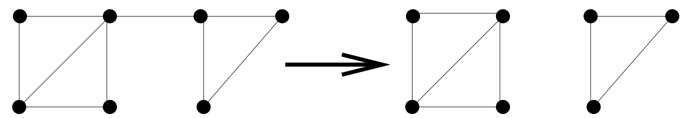

Figure 2: A graph with a bridge, and with the bridge deleted

We say that G is *bridgeless* if it does not contain a cut edge. So a graph with a cycle double cover must be bridgeless. The **Cycle Double Cover** conjecture says that this necessary condition is also sufficient.

The Cycle Double Cover Conjecture:

Every bridgeless graph has a cycle double cover.

Where did this conjecture come from? It came from thinking about special kinds of graphs called *planar graphs*. We say that a graph is *planar* if you can draw it in the plane (for example, on a piece of paper) with no edges crossing. At first glance, it looks like the graph H shown in Figure 1 is not planar. You can see that in our drawing of H, the edge $\{a, e\}$ crosses the edge $\{b, d\}$. But we can redraw H so that no edges cross, which shows that H is a planar graph. See Figure 3.

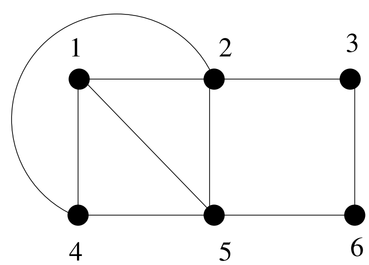

Figure 3: A planar graph drawing of the graph H

Suppose that we have a planar graph G, drawn on a piece of paper without any edges crossing. Such a drawing is called an *embedding* of G in the plane. So Figure 3 actually shows a particular embedding of the planar graph H in the plane. Now imagine that you take a sharp pair of scissors and cut out the edges and vertices of the graph H, leaving all the white space behind. Instead of one sheet of paper, you are left with two triangular pieces, a square piece and the large outside piece. Each of these pieces is called a *face* of our embedding of H. (Different embeddings of a planar graph may produce a different set of faces.) In this way we can define the set of faces of any embedding of a planar graph. The large 'outside piece' is called the *outer face*.

Now, we can circumnavigate each face by taking a walk around the edges of G which border the face. For our planar graph H, these walks describe the following cycles in the graph:

$$abda, \qquad adea, \qquad abea, \qquad bcfeb, \qquad bcfedb$$

But this is just a cycle double cover of the graph H!

Will this method always work for any planar graph G? We just have to check that each walk around a face of the embedding of G produces a cycle. Note that a cycle should not visit any vertex more than once. What could go wrong? One possibility is that an edge is used twice in the walk around a face. Since G is planar, the only way that this can happen is when G has a cut edge. For example, consider the graph on the left in Figure 3, above. The walk around the outer face will use the cut edge twice. But we can just ignore this case, since we know that a graph with a cut edge does not have a cycle double cover.

Another possibility is that a vertex may be visited more than once in some walk (but no edge is used more than once). In this case, you can split the walk into two or more cycles. For example, consider the the embedding of the planar 'bowtie' graph shown in Figure 4. Here the walk around the outer face is given by $abcdbea$. Since

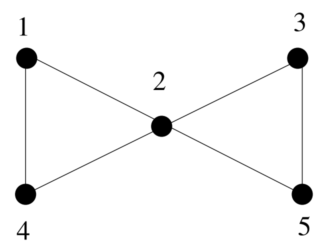

Figure 4: A "bowtie" graph

vertex b is visited twice, this is not a cycle. But we can split this walk into two cycles $beab$ and $bedb$, both starting at vertex b.

We have shown that any bridgeless planar graph G has a cycle double cover, obtained by walking around the faces of an embedding of G (and possibly splitting some of these walks into cycles). Thus the cycle double cover conjecture is true for planar graphs.

An easy proof?

Here is an attempt at an easy 'proof' of the cycle double cover conjecture. Start with a graph G, and form a multigraph G' by replacing each edge of G with two parallel edges. (We call G' a multigraph because it has multiple edge: strictly speaking it is not a graph.) For example, if we start with our first example graph H then we produce the multigraph H' shown in Figure 5. Notice that every vertex of H' has even degree.

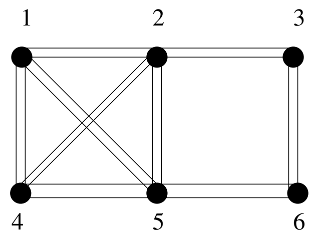

Figure 5: The multigraph H'

This is true for any multigraph G' produced by this construction. We can start removing cycles from the multigraph G' one by one. After a cycle is deleted from G', the remaining multigraph still has the property that all vertex degrees are even. So we can keep going until we have taken all the edges of G'. This covers each edge of G' exactly once, so it covers each edge of G exactly twice. This produces a cycle double cover of G, doesn't it? Have we just come up with a proof of the famous cycle double cover conjecture?

Well, not quite, because some of the 'cycles' that we are left with at the end might not really be cycles. For example, let's apply this method to the multigraph H' and see what happens if we make some bad choices. See Figure 6.

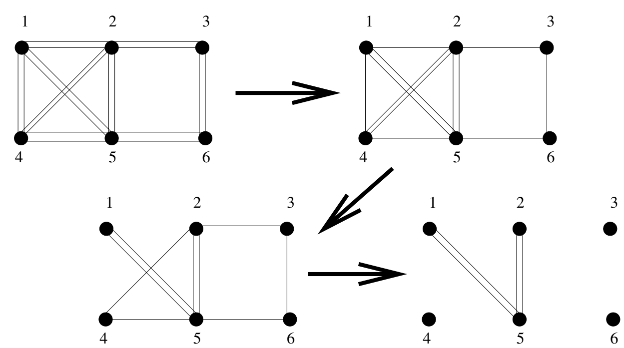

Figure 6: A bad sequence of cycle removals in H'

In the first step we remove the cycle $abcfeda$. Next we remove the cycle $abda$ in the second step. Finally, in the third step we remove the cycle $bcfedb$. But now all that remains is two copies of the edge $\{a, e\}$ and two copies of the edge $\{b, e\}$. We can use these up in the walks aea and beb, but as each of these walks only has two edges, neither of them corresponds to a cycle in H. (Remember that a cycle needs at least 3 edges.) So this easy method is not guaranteed to produce a cycle double cover.

Of course, for our graph H' we could choose the cycles to remove in a much better way, and produce the cycle double cover of H which we found earlier. But noone has been able to *prove* that a good choice of cycles always exists for an *arbitrary* bridgeless graph. Many different approaches have been tried to prove the cycle double cover conjecture and so far, none have worked.

Maybe the conjecture is false?

Of course, if the cycle double cover conjecture is so hard to prove, perhaps that means that it is false. So now let's assume that the cycle double cover is false. Then there must be a *smallest* graph for which the conjecture fails. This graph is called a *minimal counterexample*. It is a bridgeless graph with no cycle double cover and with the smallest number of edges possible. We know quite a bit about the structure of a minimal counterexample, if one exists. To describe one we must introduce some more terminology.

A *proper edge colouring* of a graph G is formed by assigning a colour to each edge of G so that all edges which meet at a common vertex are coloured with different colours. As an example, consider Figure 7 which shows a proper edge colouring of the graph H with 4 colours. The colours are indicated by the labels **1**, **2**, **3**, **4**. You can see that edges $\{a, b\}$ and $\{d, e\}$ have colour **1**, edges $\{a, e\}$ and $\{b, d\}$ have colour **2**, edges $\{a, d\}$, $\{b, e\}$ and $\{c, f\}$ have colour **3** while edges $\{b, c\}$ and $\{e, f\}$ have colour **4**. Notice that it is

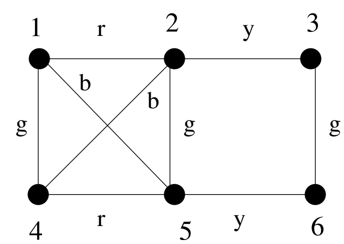

Figure 7: A proper edge colouring of H with 4 colours

not possible to properly colour the edges of H with 3 colours, since there is a vertex in H with degree 4, requiring 4 distinct colours to appear on those edges. So this edge colouring uses the minimum possible number of colours. Also, if we changed the colour of the edge ad to colour **2**, say, then the edge colouring is no longer proper since there would be two edges coloured with colour **2** which meet at vertex a.

What has this got to do with the cycle double cover conjecture? Well, Jaeger proved in 1975 that any minimal counterexample to the cycle double cover conjecture is a *snark*; that is, a connected, bridgeless cubic graph which cannot be properly edge coloured with three colours. Remember, a cubic graph is one in which every vertex has 3 neighbours. So at least 3 colours are required for a proper edge colouring of any cubic graph, but a snark needs more than 3 colours.

The oldest snark is the *Petersen graph*, discovered in 1891. It is shown in Figure 8. For a long time, noone knew whether there were any other snarks at all.

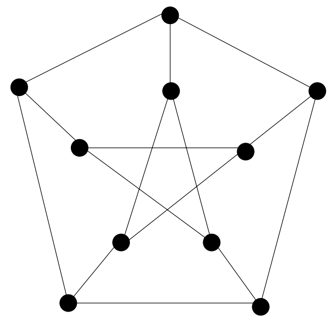

Figure 8: The Petersen graph

Exercise: Check that the Petersen graph is a snark. (Well, it is fairly easy to see that it is connected, bridgeless and cubic. It remains to convince yourself that you cannot properly colour the edges of the Petersen graph with 3 colours.) Easier: find a proper colouring of the edges of the Petersen graph with 4 colours.

Two more examples of snarks, both with 18 vertices, were found by a Croatian mathematician called Blanuša in 1946. These snarks have appeared on Croatian postage stamps! Figure 9 shows the two Blanuša snarks and the so-called 'flower' snark with 20 vertices. A few more snarks were discovered over the years. For example, in 1973

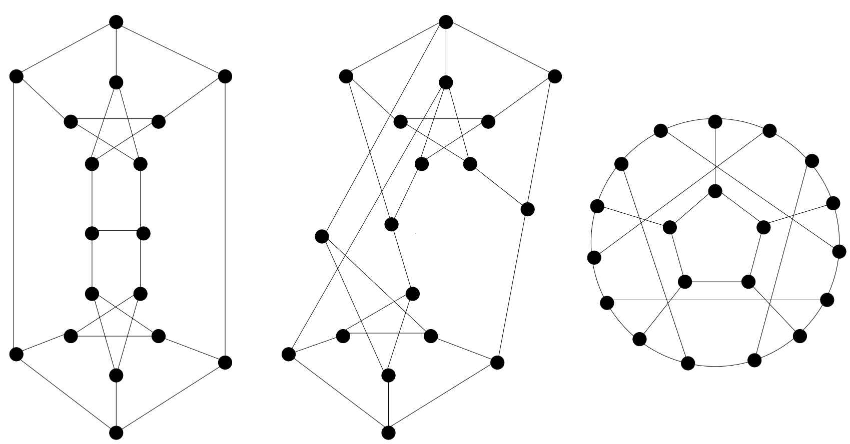

Figure 9: The two Blanuša snarks and the flower snark

George Szekeres found the *Szekeres snark* which has 50 vertices. Finally Isaacs produced two *infinite* families of snarks in 1975.

Since we have these snarks, doesn't this mean that the conjecture is false? No. It has been shown that all of the known snarks have a cycle double cover, so none of them are a counterexample.

There are some strange words used in graph theory, but ‘snark’ must be one of the strangest. The name was suggested by Martin Gardner in 1975, after the creature from the poem ‘The Hunting of the Snark’ by Lewis Carroll. In this poem, the snark is a fabled beast who is very elusive. Since examples of these graphs were so hard to find for so long, Gardner thought that they deserved the name ‘snark’.

The cycle double cover conjecture is still open and still motivating research in graph theory today. Attempts at a proof have led to many other interesting questions and results. People are still interested to see whether the conjecture is true or false. The hunting of the (minimal counterexample) snark continues.