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Binomial coefficients and related counting numbers

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We need counting in our daily life. Collecting cash from the supermarket, checking the bill at a restaurant, counting the number of place settings at a dinner party. . . This sort of counting is pretty easy because we can count one by one. However, in some particular but not unusual situations, even if the collection of objects is very easy to describe it may be difficult to count. For example: *How many ways are there to give 30 books to 7 friends?* Let's hold on to the question and discuss it later.

In this article, starting from the binomial coefficients, we'll look at some of the collections and the kinds of numbers which arise when we count them. We call them counting numbers.

Binomial coefficients and their recursive definition

The most fundamental numbers that arise in counting collections of objects in a set are the binomial coefficients. This is because in counting a set, we often count subsets, subsets of a given size and combinations of subsets. Thus in this section we'll introduce the binomial coefficients and then introduce a very similar number—Stirling numbers in the next section.

Problem: How many distinct subsets which have k elements are there in a finite set N ?

Definition 0.1 *For a set N, if $|N| = n$ is the number of elements in N, and K is a subset of N with $|K| = k$ being the number of elements in K, we say that N is an n-**set** and that K is a k-**subset** of N.*

Definition 0.2 *The number of k-subsets of an n-set is denoted by the **Binomial Coefficient** $\binom{n}{k}$ or $C(n, k)$*

The solution to the problem above is just the same as calculating $C(n, k)$. We argue as follows.

First we think of the number of ordered k-tuples formed from elements of N. The first component of the ordered k-tuple can be filled by one of n elements of N, the second by any of the remaining $n-1$ elements, etc. The kth component can be picked from the last remaining $n-k+1$ elements of N. Hence, the total number of ordered k-tuples is

$$n(n-1)(n-2)\ldots(n-k+1) = \frac{n!}{(n-k)!}$$

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where $n! = n(n-1)\dots 1$ is the factorial of n with the convention that $0! = 1$.

On the other hand, let the final answer of the above problem be M. From each of M increasing sequences, we can form $k!$ ordered k-tuples. Hence the number of all ordered k-tuples whose components are elements of N equals $k! \cdot M$. Therefore, $k! \cdot M = \frac{n!}{(n-k)!}$ So $M = \frac{n!}{k!(n-k)!}$ that is to say, the Binomial Coefficient $C(n,k) = \frac{n!}{k!(n-k)!}$. Obviously, $C(n,0) = C(n,n) = 1, \forall n \geq 0$. Normally, we write as follows:

$$C(n,k) = \begin{cases} \frac{n!}{k!(n-k)!}, & 0 \leq k \leq n \\ 0, & k < 0 \text{ or } k > n \end{cases}.$$

If we calculate $C(n,k)$ for the first several values of n and k, we can form a table as follows:

n\k	0	1	2	3	4
0	1	0	0	0	0
1	1	1	0	0	0
2	1	2	1	0	0
3	1	3	3	1	0
4	1	4	6	4	1

As mentioned before, the counting numbers $C(n,k)$ are called binomial coefficients. Now we see where this name came from. We obtain the same numbers that appear in the fully expanded powers of a binomial:

$$(a+b)^0 = 1$$

$$(a+b)^1 = a+b$$

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

$$\vdots$$

In general Newton's binomial formula holds:

$$(a+b)^n = \sum_{k=0}^{n} C(n,k) a^{n-k} b^k$$

This can be proved by following combinatorial thinking: expanding the expression $(a+b)^n$ yields a sum whose terms all have the form $A_k a^{n-k} b^k$, $0 \leq k \leq n$, where the number A_k are called the *binomial coefficients*. We next show that A_k equals the number of k-subsets of an n-set. In the equality: $\underbrace{(a+b)(a+b)\cdots(a+b)}_{n} = \sum_{k=0}^{n} A_k a^{n-k} b^k$ the term $a^{n-k}b^k$ appears as many times as there are ways of selecting k letters b from n boxes. The order of boxes is not important because the order is irrelevant in multiplication. It always yields b^k. Therefore the binomial coefficients equal the number of k-subsets of an n-set. In other words $A_k = C(n,k)$ which justifies the name binomial coefficients.

In particular, taking $a = 1,\ b = x$, we obtain the polynomial identity:

$$(1+x)^n = \sum_{k=0}^{n} C(n,k)x^k.$$

As we learned before, $C(n,0) = C(n,n) = 1, \forall n \geq 0$

Furthermore, the binomial coefficients satisfy the following identity (called Pascal's formula):

$$C(n,k) = C(n-1,k-1) + C(n-1,k), \qquad 1 \leq k \leq n-1$$

We prove it as follows. Among n elements of the initial set N we select one and call it x. All k-subsets of N are divided into two disjoint groups according to whether or not these contain x. There are $C(n-1,k-1)$ subsets containing x, because besides x we are free to pick $k-1$ elements from the $(n-1)$-set $N - \{x\}$. Similarly, if the subsets don't contain x, we can only pick all k elements from the $n-1$ remaining, i.e. from the $(n-1)$-set $N - \{x\}$. So there are $C(n-1,k)$ subsets not containing x. Then we get Pascal's formula, which is a kind of recurrence relation.

The binomial coefficient $C(n,k)$ can also be defined in another way using a recursive definition where:

$$\begin{cases} C(n,0) = C(n,n) = 1, & \forall\ n \geq 0. \\ C(n,k) = C(n-1,k-1) + C(n-1,k), & 1 \leq k \leq n-1. \end{cases}$$

From the above definition, we can form a **Pascal's triangle** (the first five rows):

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \\ & & & & \vdots & & & & \end{array}$$

Now let's go back to look at the problem offered at first: *How many ways are there to give 30 books to 7 friends?* This question is not precise because it doesn't specify whether or not the books are the same, and whether or not every friend receives at least one book. Let us consider a few possibilities:

1. Let the books differ. Each of the 30 books can go to one of the 7 sets, so according to the Product Rule, the total number of possibilities is 7^{30}.

2. Let us assume again that all the books differ, but each friend receives at least one book. First give one book to each friend. This can be done in $30 \cdot 29 \cdot 28 \cdot 27 \cdot 26 \cdot 25 \cdot 24$ ways. The remaining 23 books can be given in 7^{23} ways, making the total: $30 \cdot 29 \cdot 28 \cdot 27 \cdot 26 \cdot 25 \cdot 24 \cdot 7^{23}$.

3. Now assume the books are all the same. We draw 30 circles to represent the 30 books and 6 vertical lines to represent that the circles on the left and right of the lines are given to different friends. Every such diagram uniquely represents one arrangement of books. That is to say we select 6 places from 36 places to put lines and the rest to put circles. According to the definition of the binomial coefficient, the number of diagrams with lines and circles is $C(36, 6)$.

4. Let the books be the same but make sure that each friend receives at least one book. We use the same method as before to draw the diagrams but notice that in order to satisfy the condition we can only draw lines between two circles. So there are only 29 places to put 6 lines. Therefore the answer is $C(29, 6)$.

Stirling numbers and their recursive definitions

In the problem we discussed above there is an expression: $30 \cdot 29 \cdot 28 \cdot 27 \cdot 26 \cdot 25 \cdot 24$, which represents the number of ways of giving one book to 7 friends. This kind of expression is useful in counting collections of objects. Generally, we have the following definition.

Definition 0.3 *The expression $x(x-1)(x-2)\dots(x-n+1)$ is called a* ***falling factorial of length n****, denoted by $(x)_n$ $(n \in \mathbb{N}_0)$ with the convention $(x)_0 = 1,$ for all x.*

Recall the polynomial identity: $(1+x)^n = \sum_{k=0}^{n} C(n,k)x^k$. We now express the falling factorials $(x)_n$ in a similar representation.

Definition 0.4 *The* ***Stirling numbers of the first kind*** *$s(n,k)$ are defined as the connection coefficients in the polynomial identity:*

$$(x)_n = \sum_{k=0}^{n} s(n,k)x^k \quad (n, k \in \mathbb{N}_0)$$

with the convention: $s(0,0) = 1,\ s(n,0) = 0$ for $n \geq 1$.

We'll see the counting meaning of $s(n,k)$ in the next section.

Definition 0.5 *A* ***partition*** *of a finite set S into k parts (or blocks) is a collection of subsets $B_1, B_2, \dots B_k$ of S such that: 1) $|B_i| \neq 0,\ \forall i$ 2) $B_i \cap B_j = \varnothing,\ \forall i \neq j$ 3) $B_1 \cup \dots \cup B_k = S$*

Definition 0.6 *We write $S(n,k)$ for the number of partitions of an n-set into k blocks. $S(n,k)$ is called a* ***Stirling number of the second kind****.*

Stirling numbers (of the second kind) arise when we count partitions. They are perhaps the most common numbers occurring in counting problems, after binomial coefficients. They are harder to deal with than binomial coefficients (we have no simple formula to calculate them directly) but we do have useful recurrence relations similar to Pascal's formula. We'll also see that even the method of deducing the relations is similar to that of binomial coefficients.

Proposition: For all $n, k \geq 0,\ s(n,k),\ S(n,k)$ satisfy the following recurrence relations:

i) Stirling number of the first kind

$$\begin{cases} s(n,0) = 0, & \forall n \geq 1 \\ s(n,n) = 1, & \forall n \geq 0 \\ s(n,k) = s(n-1,k-1) - (n-1)s(n-1,k), & 1 \leq k \leq n-1. \end{cases}$$

ii) Stirling number of the second kind

$$\begin{cases} S(n,0) = 0, & \forall n \geq 1 \\ S(n,n) = 1, & \forall n \geq 0 \\ S(n,k) = S(n-1,k-1) + kS(n-1,k), & 1 \leq k \leq n-1. \end{cases}$$

Proof: The initial conditions can be obtained directly from the definitions of the Stirling numbers.

i) For all $n \in \mathbb{N}_0$, we have $(x)_n = x(x-1)\dots(x-n+2)(x-n+1) = (x)_{n-1}(x-n+1) = x(x)_{n-1} - (n-1)(x)_{n-1}$ according to the identity in the definition of $s(n,k)$, then

$$\sum_{k=0}^{n} s(n,k)x^k = x\sum_{k=0}^{n-1} s(n-1,k)x^k - (n-1)\sum_{k=0}^{n-1} s(n-1,k)x^k$$
$$= \sum_{k=0}^{n-1}\{s(n-1,k)x^{k+1} - (n-1)s(n-1,k)x^k\}.$$

Comparing the coefficients of x^k in this identity, we get:

$s(n,k) = s(n-1,k-1) - (n-1)s(n-1,k), \quad 1 \leq k \leq n-1.$

ii) Considering an n-set N, let us count the partitions of this set into k blocks. By definition, there are $S(n,k)$ such partitions. But we can count them in an alternative way, by counting those partitions that have one element, say x, as a block and those that don't.

If $\{x\}$ is a block of the partition, we need to divide the $(n-1)$-set $N - \{x\}$ into $k-1$ blocks and there are $S(n-1,k-1)$ ways of doing this.

If $\{x\}$ is not a block, then x must be contained in a block with at least one other element of N. There are $S(n-1,k)$ ways of partitioning $(n-1)$-set $N - \{x\}$ into k blocks and x may lie in any one of these blocks. Hence there are a total of $kS(n-1,k)$ ways in which N can be partitioned into k blocks without $\{x\}$ as a block.

Putting all this together we obtain: $S(n,k) = S(n-1,k-1) + kS(n-1,k), \quad 1 \leq k \leq n-1.$ $\square$

In fact, we can define $s(n,k),\ S(n,k)$ by the above recurrence relations.

These yield the following Pascal-type triangles for Stirling numbers (the first six rows):

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 0 & & 1 & & & & \\
 & & & 0 & & -1 & & 1 & & & \\
 & & 0 & & 2 & & -3 & & 1 & & \\
 & 0 & & -6 & & 11 & & -6 & & 1 & \\
0 & & 24 & & -50 & & 35 & & -10 & & 1 \\
 & & & & & \vdots & & & & &
\end{array}
\qquad
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 0 & & 1 & & & & \\
 & & & 1 & & 0 & & 1 & & & \\
 & & 0 & & 1 & & 3 & & 1 & & \\
 & 0 & & 1 & & 7 & & 6 & & 1 & \\
0 & & 1 & & 15 & & 25 & & 10 & & 1 \\
 & & & & & \vdots & & & & &
\end{array}
$$

Stirling numbers of the first kind   Stirling numbers of the second kind

Partition numbers and Bell numbers

Recall the example we discussed before: *How many ways are there to give 30 books to 7 friends?* In the fourth case, we let the books be the same and make sure that each friend receives at least one book. In fact, the problem is equivalent to counting all positive integer solutions of the equation: $n_1 + n_2 + \cdots + n_7 = 30$. We call them (ordered) 7-partitions of 30. Sometimes we use the term *number partition* to distinguish the partitions of sets introduced earlier.

Definition 0.7 *A* ***partition*** *of a positive integer n is a representation of n as a sum of one or more positive integers. A k-partition denotes a partition of n with exactly k parts.*

Definition 0.8 *We write $P(n,k)$ for the number of unordered k-partitions of n. $P(n,k)$ is called a* ***Partition number****.*

Proposition: The partition number $P(n,k)$ has the following recurrence relations: for all $n \geq k \geq 1$.

$$
\begin{cases} P(n,1) = P(n,n) = 1 \\ P(n,k) = P(n-1,k-1) + P(n-k,k) \end{cases} .
$$

Proof: The initial values P(n,1)=P(n,n)=1 are clear. All the k-partitions can be divided into two disjoint groups according to whether or not they have 1 as a summand. If 1 is a summand, there are $P(n-1,k-1)$ partitions of $n-1$ into $k-1$ parts. If 1 is not a summand, first we can put 1 to each of the k parts, and then partition the remaining $n-k$ into k parts. So the number of partitions is $P(n-k,k)$. Therefore, the total number of unordered k-partitions of n equals $P(n-1,k-1) + P(n-k,k)$. $\square$

Definition 0.9 *The* ***Bell number*** *B_n is defined as the number of partitions of an n-set.*

From the definition, since the Stirling number of the second kind $S(n,k)$ denotes the number of k-partitions of an n-set, we easily get: $B_n = \sum_{k=0}^{n} S(n,k)$. The Bell numbers have the following recurrence relation:
Proposition: For $n \geq 1$, then

$$\begin{cases} B_0 = 1 \\ B_n = \sum_{k=0}^{n-1} C(n-1,k) B_k \end{cases}$$

Proof: The initial value is clear. Partition $\{1, 2, \cdots, n\}$ into $\mathbb{B} = \{N_1, \cdots, N_n\}$. Suppose $n \in N_1$. Let $X = N_1 - \{n\}$ and $|X| = k$. $\mathbb{B}$ can be formed by choosing k elements for X in $C(n-1,k)$ ways and partition the remaining $n-1-k$ elements in B_{n-1-k} ways.

Hence, $B_n = \sum_{k=0}^{n-1} C(n-1,k) B_{n-1-k} = \sum_{k=0}^{n-1} C(n-1,k) B_k$. □

Generating functions

In the course of our investigation of counting problems we have encountered many sequences of numbers $\{f(n)\}$ depending on an integral parameter n:

- The binomial coefficients $f(n) = C(m,n),\ n \in \mathbb{N}$ where m is a fixed positive integer.
- The Bell numbers $f(n) = B_n,\ n \in \mathbb{N}$.

Our goal is now to find the solution $f(0), f(1), \ldots$ in a closed form instead of having to evaluate each term $f(n)$ individually. In this section we introduce one of the most successful devices for studying a sequence of numbers, by treating them as coefficients in *a formal power series* $\sum_{n=0}^{\infty} f(n) x^n$ and developing methods to compute this series.

Definition 0.10 *If* $f(0), f(1), \ldots = \{f(n)\}_{n=0}^{\infty}$ *is a sequence, we call*

$$g(x) = f(0) + f(1)x + f(2)x^2 + \ldots = \sum_{n=0}^{\infty} f(n) x^n$$

the ***generating function*** *for the coefficients* $f(n)$.

Note: Here x is not a variable, x^n acts as a 'place holder' for $f(n)$: that is to say, x^n merely marks the place where $f(n)$ is written. So x may be replaced by any other symbol.

We know the binomial formula: $(1+x)^n = \sum_{k=0}^{n} C(n,k) x^k$. So, $g(x) = (1+x)^n$ is the generating function for the binomial coefficients $C(n,k)$ for a fixed n.

From the definition of the Stirling number of the first kind:

$$(x)_n = \sum_{k=0}^{n} s(n,k)x^k \quad (n, k \in \mathbb{N}_0).$$

We know $(x)_n = x(x-1)\ldots(x-n+1)$ is the generating function for the Stirling number of the first kind $s(n,k)$ for a fixed n.

Proposition: $x^n = \sum_{k=0}^{n} S(n,k)(x)_k$

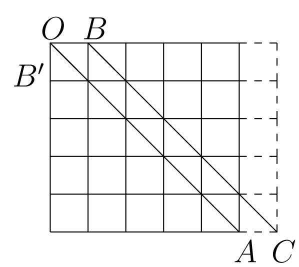
Proof: Let N be an n-set, X be an x-set. We count the collection of functions $f : N \to X$ in two ways.

First, the number of functions is x^n. Secondly, each $f : N \to X$ is subjective onto a unique subset Y of X with $|Y| \leq n$. If $|Y| = k$, there are $k!S(n,k)$ such functions. There are $C(x,k)$ choices of subsets Y of X with $|Y| = k$.

Hence, $x^n = \sum_{k=0}^{n} k!S(n,k)C(x,k) = \sum_{k=0}^{n} S(n,k)(x)_k \quad \square$

Catalan numbers

The final counting numbers that we consider in this article are the so-called Catalan numbers. To motivate these numbers we first consider the following problem.



An ant is to go from the top left corner to the bottom right corner along the lines of a square $n \times n$ grid (such as the one shown in the figure). Assume it has only two methods of walking: go right or go down. Question: 1) How many paths does the ant have? 2) If we draw a diagonal line between the start and the destination, and demand the ant can touch it but should not pass it, determine the number of paths.

Solution: 1) According to the condition, the ant must go right and go down for n steps (the edge of the small square called a step) in total to complete the trip. If we regard the $2n$ steps which the ant takes as a set, n elements of it should be chosen to be right steps and the other n elements to be down steps. As we learned before, the number of the ways is just the binomial coefficient $c(2n,n)$. So the ant has $c(2n,n) = \frac{(2n)!}{n!n!}$ ways.

2) Let's consider the ways of passing the diagonal. We mark the start as O and the destination as A. First add $n \times 1$ squares beside the right edge of the grid. Mark the points one right step from O and A as B and C respectively. If we join B and C we get a diagonal parallel with OA.(see above graph). According to the translation, the ways from O to A and those from B to C correspond one-to-one. A path from O to A which

passes OA corresponds to a path from B to C which touches OA. We can reflect such a path to the other side of OA, and get a path from the point one down step from O, marked as B', to C. From B' to C, the ant has to go right for $n+1$ steps and go down for $n-1$ steps, so there are $\frac{(2n)!}{(n+1)!(n-1)!}$ ways. That's exactly the ways from O to A which passes OA. As we discussed in 1), the total number of ways from O and A is $\frac{(2n)!}{n!n!}$.

Therefore, the result is

$$\frac{(2n)!}{n!n!} - \frac{(2n)!}{(n+1)!(n-1)!} = \frac{1}{n+1} \cdot \frac{(2n)!}{n!n!} = \frac{1}{n+1} c(2n, n). \square$$

The number $\frac{1}{n+1} c(2n, n)$ is called the n-th **Catalan number**. In this article we are not going to discuss it in detail. If you're interested in them and would like to learn more, please refer to related books.

Further Reading

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