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History of Mathematics: Solving Cubic Equations

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In my last column, I showed the way in which the solution of cubic equations led to the introduction of complex numbers. Here I will concentrate on the solution of cubic equations themselves. Some of the early history was given in my previous account, which included a very brief summary of the contributions of Cardano and his associates Fior, Fontana (Tartaglia), del Ferro and Ferrari.

Modern opinion usually attributes the solution of the cubic to Tartaglia, although it may well be that del Ferro was the actual discoverer. The task is to find a formula giving a solution to the equation

$$a_0x^3 + a_1x^2 + a_2x + a_3 = 0, \tag{1}$$

where a_0, a_1, a_2, a_3 are constants (generally taken as real numbers). If a_0 is zero, the equation reduces to a quadratic; if not then we may divide throughout by it and so reach a form in which $a_0 = 1$. This is usually assumed to have already been done, and this convention is adopted here. A further straightforward simplification is also possible, as explained in my earlier column (*Parabola*, Vol.41, No.2, 2005, p13). We can always rewrite the equation in a form in which $a_1 = 0$. This gives the so-called 'reduced form', which here (following most other discussion) will be regarded as the standard. That is to say, the general cubic may be written in the form:

$$x^3 + a_2x + a_3 = 0. \tag{2}$$

The next point to note is that the cubic equation necessarily possesses at least one real root. This is because when x is a sufficiently large negative number, the left-hand side of equation (2) is necessarily negative, whereas if x is a sufficiently large positive number, then that left-hand side is necessarily positive. It follows that somewhere between these extremes there must be a (positive or negative) number for which it is zero, and this will give us a (real) solution of equation (2). A more detailed analysis tells us that there will be either one or three real roots, with special subcases in which two or even three turn out to be equal.

Nowadays we say that there are always three roots, of which at least one is real, but the other two may be complex. This was also detailed in my previous column. However, in that column, we contended that the use of complex numbers derives from the work of Bombelli, who came somewhat later than the solvers mentioned above. In fact the paradoxical situation was reached that if all the three roots are *real*, then we need to use complex numbers to find them.

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However, with the benefit of our twenty-first century knowledge, we can say that in the case where there are two complex roots, then the formula given in the last column can always supply us with the third, the one that is real, and once we know this, then the other (complex) ones can be found. Take as an example the cubic

$$x^3 + x - 2 = 0.$$

It is not hard to see that $x = 1$ is a solution of this equation. (We can get this from 'the formula', but it is a hard slog, and simple trial and error works better!) However, armed with this knowledge, we can factor the left-hand side into the form $(x-1)(x^2 + ax + b)$. Now expand this expression, compare the coefficients, and so find that $a = 1$ and $b = 2$. The second factor is $x^2 + x + 2$, and if we set this equal to zero, we discover the other two roots, $x = \dfrac{-1 \pm i\sqrt{7}}{2}$.

Before proceeding, however, it is best to note that the Italian group listed above were not the first mathematicians to contribute to the relevant theory. Some 500 years earlier, Omar Khayyám, today better remembered as a poet, had stated that while quadratic equations were solvable by ruler and compass geometric constructions, cubics were not, but could be solved by means of geometric constructions involving conic sections.

In my column in *Function* for August 2002, I gave details of one such construction involving a parabola; other such geometric solutions are given by the historians of Mathematics Howard Eves [in an article in *The Mathematics Teacher* (1958)] and Julian Coolidge [in Chapter II of his book *The Mathematics of Great Amateurs*].

We tend to overlook Omar's contributions for a number of reasons. One is doubtless a certain eurocentricity of outlook, but there is also the big difference that his solutions were geometric constructions, whereas the later Italian approaches were algorithmic. Omar also considered only positive solutions, although his methods could be modified to apply also to negative ones.

Both Omar and the later Italians distinguished certain special cases and treated them as different; it was only later and gradually that we came to learn that equation (2) is a general form, and that the big distinction is that between the case of one real root and that of three.

Let us now look at equation (2) in more detail. I here follow J.N. Crossley's account in *The Emergence of Number*. We know that

$$(u-v)^3 = u^3 - 3uv(u-v) - v^3.$$

So put $x = u - v$, and reach

$$x^3 + 3uvx = u^3 - v^3. \tag{3}$$

We now seek to reconcile equations (2) and (3). They are exactly the same equation if

$$3uv = a_2 \qquad \text{and} \qquad u^3 - v^3 = -a_3. \tag{4}$$

This approach is today attributed to del Ferro (who also may have anticipated Cardano in realising that equation (2) is a general form.

It tells us that u^3 and $-v^3$ are two quantities whose sum is $-a_3$ and whose product is $\frac{-a_2^3}{27}$. They are thus the roots of the quadratic equation

$$y^2 + a_3 y - \frac{a_2^3}{27} = 0. \tag{5}$$

So now all we have to do to solve the cubic (2) is to solve the quadratic (5). In my previous column, I presented a variation of this argument and attributed it to A.J.B. Ward. In fact, I reordered Ward's argument in order to bring out his essential point, that the process of solution is closely analogous to the method of 'completing the square' by which we solve quadratics. Ward's original approach is really the same as that of del Ferro's.

The complications that arise from this point on were all covered in my previous column, and so I won't repeat them here. Rather, I will show other approaches to the solution of the cubic, most of which proceed by reducing it to a quadratic.

I first encountered the solution of cubic equations in a textbook that was once very popular, Durrell & Robson's *Advanced Algebra*. I picked up my copy in a second-hand bookstore in 1956. It is a three-volume work and comes with a companion volume, *Advanced Trigonometry*. In Volume II of the Algebra book a method is given for solving cubics. It is attributed to Tartaglia, but this cannot be correct, as it relies on complex numbers, which came later.

It begins with an identity that reads (in the notation I am using here):

$$x^3 + 3uv - u^3 + v^3 = (x - u + v)(x - \omega u + \omega^2 v)(x - \omega^2 u + \omega v) \tag{6}$$

where $\omega = \frac{-1 + i\sqrt{3}}{2}$, a complex cube root of 1. (I leave it to readers to check the correctness of this formula).

But now make the left-hand side of the identity (6) read the same as the left-hand side of equation (2). That is to say, set

$$a_2 = 3uv \qquad \text{and} \qquad -a_3 = u^3 - v^3,$$

but these are just equations (4), which we found before. The rest of the procedure is just the same.

Perhaps the most elegant approach is an ingenious change of variable attributed by some to the English mathematician Thomas Harriot (1560-1621) and by others to François Viète (1540-1603), who was French.

Write

$$a_2 = 3b^2 \qquad \text{and} \qquad a_3 = -2c^3. \tag{7}$$

Now put

$$x = \frac{z^2 - b^2}{z} \tag{8}$$

and simplify. After some tedious, but otherwise straightforward algebra, we reach the sixth-degree equation

$$z^6 - 2c^3 z^3 = b^6. \tag{9}$$

At first thought, we might think that to change a cubic (third- degree) equation into a sixth-degree one is to go in the wrong direction, but equation (9) is of a particularly simple type. Indeed it is a quadratic in y^3, and so we can readily solve it. Application of the familiar quadratic formula gives us

$$z^3 = c^3 \pm \sqrt{b^6 + c^6}. \tag{10}$$

The approach actually echoes the earlier ones, because if we substitute $z^3 = y$ into equation (9), we reach a (rewritten) form of equation (5).

There is yet another approach, and this one is genuinely different. It stems from the work of Ehrenfried von Tschirnhaus (or Tschirnhausen, his name is spelt both ways), a minor seventeenth century mathematician. He introduced the Tschirnhaus transformation, which aims to simplify equation (2) even further. We saw in my previous column how a simple linear function replacing x by $x + h$ produced the reduced form (2) from the more general one by making $a_1 = 0$. Such a transformation can in fact be used on an equation of any degree to achieve the same effect. Tschirnhaus sought more general transformations which would also make $a_2 = 0$.

There are various possibilities, but the one most associated with his name considers setting (in an nth degree equation):

$$y = x^{n-1} + b_1 x^{n-2} + \ldots + b_{n-1}.$$

In the case of the cubic, we seek to find a new unknown y, where

$$y = x^2 + b_1 x + b_2 \tag{11}$$

and the idea is to get an even simpler equation than equation (2), but in this new variable y. In fact, it will read

$$y^3 = A_3 = \alpha^3 \quad \text{(say)}. \tag{12}$$

This new equation has the roots $y = \alpha, \omega\alpha$ and $\omega^2\alpha,$ and then it seems a simple matter to put these three values of y into equation (11), solve the resulting quadratics and so find x.

In fact the procedure is absolutely horrendous, and even computer algebra packages are of only limited help. I found a few textbooks that give some assistance. One, *An Introduction to the Theory of Equations* by the eminent historian of Mathematics, Florian Cajori, even sets out to give the application of the Tschirnhaus to the cubic (in its most general form: equation (1)), but sadly loses puff before the task is properly complete. It then sets the application to equation (2) as an exercise. Another, R. Bruce King's *Beyond the Quartic Equation* gives better approaches than does Cajori at several points of the computations, and a third, L.E. Dickson's *Elementary Theory of Equations* explains some points left hanging by King.

Thus aided, I eventually produced the application to equation (2), (Cajori's exercise). It occupied six pages of heavy algebra, even after extensive use of the computer algebra package MAPLE. I won't go into the details here. However a few comments are in order.

We do not begin by substituting equation (11) directly into equation (2), which would straightaway give us a sixth degree equation. Rather, extensive use is made of the relations between the roots of equations (2) and (12) and their coefficients. These relations are then used to find the values of b_1, b_2 and α.

One further complication is that there are two possible values of b_1, and these give rise to two possible values of α. If we also consider that there are two roots of each of the quadratics arising from equation (11), we find that there are 12 potential roots in all. One set of three is the same as another, and these three are the right ones; the other six are ring-ins!

We might wonder why this apparently awful approach would ever be used. However, when we get to higher degree equations, it becomes a sad necessity. The Italian school solved the cubic and also the quartic. (Ferrari did that.) But there progress halted. Eventually Abel and Ruffini showed that the task of solving the quintic was not possible in terms of the usual arithmetic functions. But this was not before much work had been spent in trying to do the impossible. (Even for some time *after* the publication of these proofs, there was not universal acceptance of the result.)

The story from here on is extremely complicated, and lately this area of research has been revitalised to some extent by the continuing development of Computer Algebra packages. It is indeed this story that King, in the book detailed above, is concerned to tell us. But we can come away with a sense of awe at the persistence and sheer grind put in by the nineteenth century mathematicians, who did all this work, and did it all by hand!