Parabola Volume 41, Issue 3 (2005)

Solutions to Problems 1181–1190

Q1181. Consider the following set of linear equations

$$\begin{aligned} x + 2y + z &= 1 \\ -2x + \lambda y - 2z &= -2 \\ 2x + 6y + 2\lambda z &= 3 \end{aligned}$$

where x_1, x_2, x_3 are variables and λ is a parameter. Find all values of λ for which these equations do not have a unique solution.
ANS: First write the system of equations in matrix form

$$\begin{pmatrix} 1 & 2 & 1 \\ -2 & \lambda & -2 \\ 2 & 6 & 2\lambda \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}.$$

Now evaluate the determinant of the matrix

$$\begin{aligned} \begin{vmatrix} 1 & 2 & 1 \\ -2 & \lambda & -2 \\ 2 & 6 & 2\lambda \end{vmatrix} &= \begin{vmatrix} \lambda & -2 \\ 6 & 2\lambda \end{vmatrix} - 2 \begin{vmatrix} -2 & -2 \\ 2 & 2\lambda \end{vmatrix} + \begin{vmatrix} -2 & \lambda \\ 2 & 6 \end{vmatrix} \\ &= (2\lambda^2 + 12) - 2(-4\lambda + 4) + (-12 - 2\lambda) \\ &= 2\lambda^2 + 6\lambda - 8 \\ &= 2(\lambda - 1)(\lambda + 4) \end{aligned}$$

which is zero if $\lambda = 1$ or $\lambda = -4$ so the linear system does not have a unique solution for these values of λ.
Q1182. A homicide victim was found in a room that is kept at a constant temperature of $21°C$. A body temperature measurement was made at time τ and another was made one hour later. The results were:

$$T(\tau) = 27°C \quad \text{and} \quad T(\tau + 1) = 25°C$$

where time is measured in hours. Assuming that the victim's temperature was $37°C$ just before death, determine the time of death relative to time point τ.

(You may wish to consider Newton's Law of Cooling, in which experiments show that the time rate of change of the temperature of an object is proportional to the difference between its temperature and the temperature of the surrounding medium).
ANS: This is a problem that involves Newton's Law of Cooling, where it is assumed that the body is cooled via convection to the room environment and that the body

temperature can be characterised by a single value, T, that only varies with time. With these assumptions, the governing energy balance can be written as

$$\frac{dT}{dt} = -\beta(T - T_\infty)$$

where T_∞ represents the environment temperature and β is a proportionality constant that is related to the material properties of the body and the specific heat transfer environment.

The defining differential equation above is separable, or

$$\frac{dT}{T - T_\infty} = -\beta\, dt,$$

and integrating both sides gives

$$\ln(T - T_\infty) = -\beta t + c$$

or

$$T - T_\infty = Ae^{-\beta t}.$$

From the problem statement, we know that the body temperature just before death at $t = 0$ was $T(0) = T_0 = 37°C$. Using this initial condition in our equation gives

$$\begin{aligned} T(0) - T_\infty &= Ae^{-\beta 0} \\ A &= T(0) - T_\infty \end{aligned}$$

which leads to

$$T(t) - T_\infty = (T(0) - T_\infty)e^{-\beta t}$$

or

$$\frac{T(t) - T_\infty}{T(0) - T_\infty} = e^{-\beta t}.$$

This general expression for $T(t)$ represents the desired relationship for the body temperature as a function of time after death (at $t = 0$).

Now, upon arrival on the scene at $t = \tau$, the detective in charge of the homicide investigation measured the victim's temperature to be $T(\tau) = 27°C$. One hour later a second measurement gave $T(\tau + 1) = 25°C$.

From these two measurements we can determine the proportionality constant, β, in our equation and the time of death. To do this, we evaluate the equation at the two measurement times.

At $t = \tau$, $\quad \dfrac{T(\tau) - T_\infty}{T(0) - T_\infty} = e^{-\beta\tau}$

and

at $t = \tau + \Delta t$, $\quad \dfrac{T(\tau + \Delta t) - T_\infty}{T(0) - T_\infty} = e^{-\beta(\tau + \Delta t)}$.

If we divide these two equations:

$$\frac{T(\tau) - T_\infty}{T(0) - T_\infty} \frac{T(0) - T_\infty}{T(\tau + \Delta t) - T_\infty} = \frac{e^{-\beta\tau}}{e^{-\beta(\tau+\Delta t).}}$$

Therefore, solving this expression for β gives

$$\beta = \frac{1\Delta t}{\ln} \left[\frac{T(\tau) - T_\infty}{T(\tau + \Delta t) - T_\infty} \right].$$

Now with the given data
$\Delta t = 1$ hour, $T(\tau) = 27°C$ and $T(\tau + \Delta t) = 25°C$
we have,

$$\beta = \frac{1}{1\text{hour}} \ln \left[\frac{27 - 21}{25 - 21} \right] = \ln \frac{3}{2} \approx 0.41 \text{ hour}^{-1}.$$

With this proportionality constant, we can evaluate the time of death relative to time $t = 0$. In particular, solving

$$\frac{T(\tau) - T_\infty}{T(0) - T_\infty} = e^{-\beta\tau}$$

for τ gives

$$\begin{aligned} \tau = -\frac{1}{\beta} \ln \left[\frac{T(\tau) - T_\infty}{T(0) - T_\infty} \right] &= -\frac{1}{\ln \frac{3}{2}} \ln \left[\frac{27 - 21}{37 - 21} \right] \\ &= -\frac{1}{\ln \frac{3}{2}} \ln \left[\frac{3}{8} \right] = 2.42 \text{ hours.} \end{aligned}$$

For example, if the first temperature measurement was made at 12 noon, then the death occurred at about 9:35 AM (a little under 2.5 hours before the detectives arrived on the scene).

Q1183. A bus takes 30 minutes to travel (non-stop) from terminal A to terminal B. Buses leave terminal A regularly every 2 minutes, and travel at the same speed. A car leaves terminal A simultaneously with one of the buses, and travels to terminal B at 4 times the speed of the buses. How many buses will the car overtake by the time it reaches B?

ANS: At the time the car leaves A, there are 15 buses on the road, one leaving A, one arriving at B, and 13 on the way. The car takes 30/4=7.5 minutes to reach B. During this time there are 3 more buses arriving at B, because every 2 minutes there is one bus reaching B. So the car overtakes 11 buses on the way.

Q1184. Find a polynomial of least degree and integer coefficients that has $1 + \sqrt[3]{3}$ as a root.

ANS: It is clear that $x - 1 - \sqrt[3]{3}$ is a polynomial of degree 1 that has $1 + \sqrt[3]{3}$ as a root. However, this polynomial does not have integer coefficients. If we define $P(x)$ as

$$\begin{aligned} P(x) &= (x - 1 - \sqrt[3]{3})[(x-1)^2 + \sqrt[3]{3}(x-1) + (\sqrt[3]{3})^2] \\ &= (x-1)^3 - 3 \\ &= x^3 - 3x^2 + 3x - 4, \end{aligned}$$

then $P(x)$ is a polynomial of integer coefficients having $1 + \sqrt[3]{3}$ as a root.

To prove that $P(x)$ is a polynomial of lowest degree satisfying the required conditions, we prove that if $Q(x) = a_2x^2 + a_1x + a_0$ has $1 + \sqrt[3]{3}$ as a root, where a_0, a_1, a_2 are integers, then $a_0 = a_1 = a_2 = 0$.

Since $Q(1 + \sqrt[3]{3}) = 0$ there holds

$$a_2(1 + \sqrt[3]{3})^2 + a_1(1 + \sqrt[3]{3}) + a_0 = 0.$$

This is equivalent to

$$a_2(\sqrt[3]{3})^2 + (a_1 + 2a_2)\sqrt[3]{3} + a_0 + a_1 + a_2 = 0.$$

So $\sqrt[3]{3}$ is a solution to the quadratic equation $A_2x^2 + A_1x + A_0 = 0$ where the coefficients $A_2 = a_2$, $A_1 = a_1 + 2a_2$, and $A_0 = a_0 + a_1 + a_2$ are all integers. From

$$A_2(\sqrt[3]{3})^2 + A_1\sqrt[3]{3} + A_0 = 0, \tag{1}$$

there follows

$$(A_2(\sqrt[3]{3})^2 + A_1\sqrt[3]{3} + A_0)(A_2\sqrt[3]{3} - A_1) = 0,$$

or, equivalently,

$$3A_2^2 - A_1A_0 - \sqrt[3]{3}(A_1^2 - A_2A_0) = 0.$$

By multiplying both sides by $(3A_2^2 - A_1A_0)^2 + \sqrt[3]{3}(3A_2^2 - A_1A_0)(A_1^2 - A_2A_0) + (\sqrt[3]{3})^2(A_1^2 - A_2A_0)^2$ and using $a^3 - b^3 = (a-b)(a^2 + ab + b^3)$ we obtain

$$m^3 - 3n^3 = 0, \tag{2}$$

where $m = 3A_2^2 - A_1A_0$ and $n = A_1^2 - A_2A_0$. It follows from (2) that $3|m^3$ and that $3|m$ (because 3 is a prime number). Letting $m = 3m_1$ and substituting this into (2) we deduce that $3|n$, i.e. $n = 3n_1$. Repeating the above argument we obtain $3|m_1$ and $3|n_1$, so that $3^2|m$ and $3^2|n$. Applying the same argument k times gives $3^k|m$ and $3^k|n$. So $m = n = 0$. From the definition of m and n, it is easy to deduce $A_1^3 - 3A_2^3 = 0$, which has the same form as (2). Thus $A_1 = A_2 = 0$. It follows from (1) that $A_0 = 0$. It is now easy to see that $a_0 = a_1 = a_2 = 0$.

Q1185. Find all real roots of the following simultaneous equations

$$x^5 - y^5 = 2101 \tag{3}$$

$$x - y = 1. \tag{4}$$

This problem and its answer were suggested by Julius Guest.

ANS: (suggested by J. Guest) Eliminating y between (3) and (4) provides

$$x^4 - 2x^3 + 2x^2 - x - 420 = 0. \tag{5}$$

Setting $z = x^2 - x$ in (5) yields

$$z^2 + z - 420 = 0. \tag{6}$$

Solving (6) gives $z_1 = 20$ and $z_2 = -21$. So either $x^2 - x - 20 = 0$ or $x^2 - x + 21 = 0$. The first equation has real roots $x_1 = 5$ and $x_2 = -4$. The second equation has no real roots. Hence the solutions to (3)–(4) are

$$(x, y) = (5, -4) \quad \text{and} \quad (x, y) = (-4, -5).$$

Q1186. Find all values of m such that the equation

$$\cos^4 t + m \sin^2 t + 2 = 0$$

has a solution.

ANS: Let $x = \cos^2 t$. Then $0 \le x \le 1$ and x satisfies

$$x^2 - mx + m + 2 = 0.$$

Since $x = 1$ is not a solution, the above equation is equivalent to

$$m = \frac{x^2 + 2}{x - 1} = x + 1 + \frac{3}{x - 1}.$$

Thus we will find all values of m such that the line $y = m$ cuts the graph of $y = x + 1 + \frac{3}{x-1}$ at at least one point (x, y) with $0 \le x < 1$. A sketch of $y = x + 1 + \frac{3}{x-1}$ reveals that $m \le -2$.

Q1187. At 12 o'clock the two hands of an analogue clock are together. At time t later they make an angle $\theta(t)$ (take this to be the smaller angle). How long after 12 are the two hands first in line and first together again?

ANS: The minute hand rotates at 360 degrees per hour. The hour hand rotates at 30 degrees per hour (it rotates 360 degrees in 12 hours). So t hours after 12 o'clock the minute hand rotates at $360t$ degrees and the hour hand $30t$ degrees. Therefore the angle between the two hands is $\theta(t) = 330t$, until $t = 180/330 = 6/11$, at which time the two hands are first in line. After that, the angle between them is $\theta(t) = 360 - 330t$. Thus, $\theta(t) = 0$ again when $t = 360/330 = 12/11$.

The two hands are first in line $\frac{6}{11}$ hour, and are together $\frac{12}{11}$ hours after 12 o'clock.

Q1188. If S is the area of a triangle ABC, prove that

$$S = \frac{1}{4}(a^2 \sin(2B) + b^2 \sin(2A)),$$

where a and b are the lengths of the sides opposite the angles A and B, respectively.

ANS: Let D be the point symmetrical to C about AB. Then

$$\begin{aligned}2S &= \text{area}(ADBC) = \text{area}(ADC) + \text{area}(BCD)\\ &= \frac{1}{2}AC \times AD \times \sin\angle CAD + \frac{1}{2}BC \times BD \times \sin\angle DBC\\ &= \frac{1}{2}b^2\sin(2A) + \frac{1}{2}a^2\sin(2B).\end{aligned}$$

The required formula then follows.

Q1189 Prove that if A, B, and C are three angles of a triangle then

$$\frac{\sin A + \sin B + \sin C}{\cos A + \cos B + \cos C} < 2.$$

ANS: We first note that if A and B are two angles of a triangle then

$$-\frac{\pi}{2} < \frac{A-B}{2} < \frac{A+B}{2} < \frac{\pi}{2}.$$

Therefore,

$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2} > 0. \tag{7}$$

From this there follows

$$\cos A + \cos B + \cos C > 0. \tag{8}$$

In fact, if A, B, and C are all acute angles then (8) is obviously true. Suppose one of them, say A, is obtuse. Then $\cos A + \cos B > 0$ due to (7), and $\cos C > 0$ (because C cannot be obtuse). Thus (8) also holds. We now have

$$\begin{aligned}\sin A + \sin B + \sin C &= \sin(B+C) + \sin(C+A) + \sin(A+B)\\ &= \sin B\cos C + \sin C\cos B + \sin C\cos A + \sin A\cos C\\ &\quad + \sin A\cos B + \sin B\cos A\\ &= \sin A(\cos B + \cos C) + \sin B(\cos C + \cos A)\\ &\quad + \sin C(\cos A + \cos B).\end{aligned} \tag{9}$$

Since $0 < \sin A \le 1$ and $\cos B + \cos C > 0$ (due to (7)) there holds

$$\sin A(\cos B + \cos C) \le \cos B + \cos C. \tag{10}$$

Similarly,

$$\sin B(\cos C + \cos A) \le \cos C + \cos A, \tag{11}$$

and

$$\sin C(\cos A + \cos B) \le \cos A + \cos B. \tag{12}$$

It follows from (9)–(12) that

$$\sin A + \sin B + \sin C \le 2(\cos A + \cos B + \cos C),$$

implying (due to (8))

$$\frac{\sin A + \sin B + \sin C}{\cos A + \cos B + \cos C} \le 2.$$

The equality occurs when it occurs in each of (10)–(12), i.e. when $A = B = C = \pi/2$, which is impossible. The required inequality is proved.

Q1190 Prove that for any positive integer n,

$$\frac{1}{2\sqrt{1}} + \frac{1}{3\sqrt{2}} + \frac{1}{4\sqrt{3}} + \cdots + \frac{1}{(n+1)\sqrt{n}} < 2.$$

ANS: For any $k > 0$ there holds

$$\begin{aligned}\frac{1}{(k+1)\sqrt{k}} &= \frac{\sqrt{k}}{k(k+1)} \\ &= \sqrt{k}\left(\frac{1}{k} - \frac{1}{k+1}\right) \\ &= \sqrt{k}\left(\frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}}\right)\left(\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}}\right) \\ &= \left(1 + \frac{\sqrt{k}}{\sqrt{k+1}}\right)\left(\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}}\right) \\ &< 2\left(\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}}\right).\end{aligned}$$

Therefore,

$$\begin{aligned}&\frac{1}{2\sqrt{1}} + \frac{1}{3\sqrt{2}} + \frac{1}{4\sqrt{3}} + \cdots + \frac{1}{(n+1)\sqrt{n}} \\ &< 2\left(\frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}}\right) + 2\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}}\right) + 2\left(\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}}\right) + \cdots + 2\left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right) \\ &= 2\left(\frac{1}{\sqrt{1}} - \frac{1}{\sqrt{n+1}}\right) \\ &< 2.\end{aligned}$$