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Games with Logarithms

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It is well-known that

$$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - + \cdots \text{ for } 0 \le x \le 1,$$

where the $- + \cdots$ notation is used to indicate that successive terms are alternately subtracted and added.

I gave a proof of this in an earlier article "The harmonic series and Euler's constant" (*Parabola*, Vol 40, No 1, 2004).
In the same way, one can prove that

$$\log_e(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots \text{ for } 0 \le x < 1.$$

Subtracting the second equation from the first gives

$$\log_e\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right) \text{ for } 0 \le x < 1.$$

If n is an integer greater than 1, we can put $x = \dfrac{1}{n}$, and get

$$\log_e\left(\frac{n+1}{n-1}\right) = 2\left(\frac{1}{n} + \frac{1}{3n^3} + \frac{1}{5n^5} + \cdots\right).$$

Thus, if we put $n = 2,\ 3$ and 4 in turn, we find

$$\log_e 3 = 2\left(\frac{1}{2} + \frac{1}{3\times 2^3} + \frac{1}{5\times 2^5} + \cdots\right),$$
$$\log_e 2 = 2\left(\frac{1}{3} + \frac{1}{3\times 3^3} + \frac{1}{5\times 3^5} + \cdots\right),$$
$$\log_2 \frac{5}{3} = 2\left(\frac{1}{4} + \frac{1}{3\times 4^3} + \frac{1}{5\times 4^5} + \cdots\right).$$

Let us write

$$S(n) = 2\left(\frac{1}{n} + \frac{1}{3n^3} + \frac{1}{5n^5} + \cdots\right).$$

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We have

$$\log_e 3 = S(2),$$
$$\log_e 2 = S(3),$$
$$\log_e \frac{5}{3} = S(4),$$

so

$$\log_e 5 = S(2) + S(4).$$

The game we are going to play was suggested to me by Dr. Peter Donovan, who recently retired from UNSW, and it is, to find the logarithms of the prime numbers as linear combinations of $S(n_1),\ S(n_2),\ \ldots$, where $n_1,\ n_2,\ \ldots$ are large integers, for which the series $S(n_1),\ S(n_2),\ \ldots$ converge rapidly. If we put $n = 5$, we obtain

$$\log_e \frac{3}{2} = S(5),$$

so

$$\log_e 3 = S(3) + S(5),$$

which is better than $\log_e 3 = S(2)$.
Putting $n = 6$ gives

$$\log_e \frac{5}{4} = S(6),$$

so

$$\log_e 5 = 2S(3) + S(6),$$

better than $\log_e 5 = S(2) + S(4)$.
So far, our records are

$$\log_e 2 = S(3),$$
$$\log_e 3 = S(3) + S(5),$$
$$\log_e 5 = 2S(3) + S(6).$$

Putting $n = 7$ gives

$$\log_e \frac{8}{6} = \log_e \frac{4}{3} = S(7),$$

or

$$2\log_e 2 - \log_e 3 = S(7),$$

so

$$
\begin{aligned}
\log_e 2 &= \frac{1}{2}\log_e 3 + \frac{1}{2}S(7) \\
&= \frac{1}{2}S(3) + \frac{1}{2}S(5) + \frac{1}{2}S(7) \\
&= \frac{1}{2}\log_e 2 + \frac{1}{2}S(5) + \frac{1}{2}S(7),
\end{aligned}
$$

so

$$\log_e 2 = S(5) + S(7)$$

and

$$
\begin{aligned}
\log_e 3 &= 2\log_e 2 - S(7) \\
&= 2S(5) + S(7).
\end{aligned}
$$

Putting $n = 9$ gives

$$
\begin{aligned}
&\log_e \frac{5}{4} = S(9), \\
&\log_e 5 - 2\log_e 2 = S(9), \\
&\log_e 5 = 2S(5) + 2S(7) + S(9).
\end{aligned}
$$

At this stage, our records are

$$
\begin{aligned}
&\log_e 2 = S(5) + S(7), \\
&\log_e 3 = 2S(5) + S(7), \\
&\log_e 5 = 2S(5) + 2S(7) + S(9).
\end{aligned}
$$

From now on, I will denote records by a $(*)$ in the right margin.
Putting $n = 8$ gives

$$
\begin{aligned}
&\log_e \frac{9}{7} = S(8), \\
&\log_e 7 = 2\log_e 3 - S(8), \\
&\log_e 7 = 4S(5) + 2S(7) - S(8). \qquad (*)
\end{aligned}
$$

Notice that it looks like I have changed the rules of the game. I am now allowing the coefficients of the $S(n_i)$ to be negative integers.

Putting $n = 11,\ 17$ and 19 in turn gives

$$\begin{aligned}
\log_e 2 + \log_e 3 - \log_e 5 &= S(11), \\
2\log_e 3 - 3\log_e 2 &= S(17), \\
\log_e 2 + \log_e 5 - 2\log_e 3 &= S(19).
\end{aligned}$$

From these equations we get

$$\log_e 2 = 2S(11) + S(17) + 2S(19), \qquad (*)$$
$$\log_e 3 = 3S(11) + 2S(17) + 3S(19), \qquad (*)$$
$$\log_e 5 = 4S(11) + 3S(17) + 5S(19). \qquad (*)$$

Putting $n = 21,\ 25$ and 29 in turn now gives

$$\begin{aligned}
\log_e 11 - \log_e 2 - \log_e 5 &= S(21), \\
\log_e 13 - 2\log_e 2 - \log_e 3 &= S(25), \\
\log_e 3 + \log_e 5 - \log_e 2 - \log_e 7 &= S(29),
\end{aligned}$$

so

$$\log_e 11 = 6S(11) + 4S(17) + 7S(19) + S(21), \qquad (*)$$
$$\log_e 13 = 7S(11) + 4S(17) + 7S(19) + S(25), \qquad (*)$$
$$\log_e 7 = 5S(11) + 4S(17) + 6S(19) - S(29). \qquad (*)$$

Let us stop there, and calculate the various logarithms.
It is easy to calculate all of $S(11),\ S(17),\ S(19),\ S(21),\ S(25)$ and $S(29)$ to six or seven decimal places using just the first three terms of the series. Thus

$$\begin{aligned}
S(11) &= 2\left(\frac{1}{11} + \frac{1}{3\times 11^3} + \frac{1}{5\times 11^5} + \cdots\right) \\
&\approx 2\,(0.0909091 + 0.0002504 + 0.0000012) \\
&\approx 0.1823214,
\end{aligned}$$

and in the same way,

$$
\begin{aligned}
S(17) &\approx 0.1177831,\\
S(19) &\approx 0.1053606,\\
S(21) &\approx 0.0953102,\\
S(25) &\approx 0.0800427,\\
S(29) &\approx 0.0689929,
\end{aligned}
$$

and so

$$
\begin{aligned}
\log_e 2 &= 2S(11) + S(17) + 2S(19) \approx 0.6931471,\\
\log_e 3 &= 3S(11) + 2S(17) + 3S(19) \approx 1.0986122,\\
\log_e 5 &= 4S(11) + 3S(17) + 5S(19) \approx 1.6124379,\\
\log_e 7 &= 5S(11) + 4S(17) + 6S(19) - S(29) \approx 1.9459101,\\
\log_e 11 &= 6S(11) + 4S(17) + 7S(19) + S(21) \approx 2.3978952,\\
\log_e 13 &= 7S(11) + 4S(17) + 7S(19) + S(25) \approx 2.5649491.
\end{aligned}
$$

I encourage you to continue playing the game, and to send us some of your discoveries. In particular, it would be great if you could find logarithms of primes using values of $S(n_i)$ with all the n_i greater than 100 !

Here are the results of a few more games

$$
\begin{aligned}
4\log_e 2 - \log_e 3 - \log_e 5 &= S(31),\\
2\log_e 5 - 3\log_e 2 - \log_e 3 &= S(49),\\
4\log_e 3 - 4\log_e 2 - \log_e 5 &= S(161),
\end{aligned}
$$

so

$$
\begin{aligned}
\log_e 2 &= 7S(31) + 5S(49) + 3S(161),\\
\log_e 3 &= 11S(31) + 8S(49) + 5S(161),\\
\log_e 5 &= 16S(31) + 12S(49) + 7S(161).
\end{aligned}
$$