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Games with Inverse Tangents

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In this article I am going to assume you are a little bit familiar with the inverse tangent function, $\tan^{-1} x =$ arctan$x$. There are several properties of this function that you need to know.

Firstly $\tan^{-1} x$ is defined by the property that $\tan(\tan^{-1} x) = x$ and $\tan^{-1}(\tan x) = x$. In particular if $\tan^{-1} x = \frac{\pi}{4}$ then $\tan(\tan^{-1} x) = \tan\left(\frac{\pi}{4}\right) = 1$, and hence

$$\tan^{-1} 1 = \frac{\pi}{4}. \tag{1}$$

The following addition formulae are true for the inverse tangent function,

$$\tan^{-1} a + \tan^{-1} b = \tan^{-1}\left(\frac{a+b}{1-ab}\right) \tag{2}$$

and

$$\tan^{-1} a - \tan^{-1} b = \tan^{-1}\left(\frac{a-b}{1+ab}\right), \tag{3}$$

provided $|ab| < 1$ can be derived from the identity

$$\tan(\theta+\phi) = \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi},$$

which you may know from school. The steps leading to the addition formulae are as follows:

$$\begin{aligned} \tan^{-1}\left(\tan(\theta+\phi)\right) &= \tan^{-1}\left(\frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}\right) \\ \Rightarrow\ \theta+\phi &= \tan^{-1}\left(\frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}\right). \end{aligned}$$

Now write $\theta = \tan^{-1} a$ and $\phi = \tan^{-1} b$, then

$$\begin{aligned} \tan^{-1} a + \tan^{-1} b &= \tan^{-1}\left(\frac{\tan(\tan^{-1} a) + \tan(\tan^{-1} b)}{1 - \tan(\tan^{-1} a)\tan(\tan^{-1} b)}\right) \\ &= \tan^{-1}\left(\frac{a+b}{1-ab}\right). \end{aligned}$$

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The property of the inverse tangent function that we are most interested in here is

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} \cdots \tag{4}$$

provided $|x| \leq 1$.

Before proving (3), let me show you the game to which I wish to introduce you. Suppose $x = \dfrac{1}{n}$ where n is a large integer. Then

$$\begin{aligned}\tan^{-1} x &= \tan^{-1}\left(\frac{1}{n}\right)\\ &= \frac{1}{n} - \frac{1}{3n^3} + \frac{1}{5n^5} - + \cdots\end{aligned}$$

can be calculated quickly and accurately, since the series converges rapidly. (What I mean by this is that the terms on the right get smaller very quickly, the more so as n gets larger.)

So, for example,

$$\begin{aligned}\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right) &= \tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\times\frac{1}{3}}\right)\\ &= \tan^{-1} 1\\ &= \frac{\pi}{4},\end{aligned}$$

so

$$\begin{aligned}\pi &= 4\left(\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right)\right)\\ &= 4\left(\left(\frac{1}{2} - \frac{1}{3\times 2^3} + \frac{1}{5\times 2^5} - + \cdots\right)\right.\\ &\qquad \left.+\left(\frac{1}{3} - \frac{1}{3\times 3^3} + \frac{1}{5\times 3^5} - + \cdots\right)\right),\end{aligned}$$

which gives us a way of calculating π. (We could also have used the fact that

$$\begin{aligned}\pi &= 4\tan^{-1} 1\\ &= 4\left(1 - \frac{1}{3} + \frac{1}{5} - + \cdots\right),\end{aligned}$$

but the series on the right converges very slowly.)

Before going on with the game, let me prove (4). We know that the sum to n terms of the geometric series is

$$1 - t^2 + t^4 - t^6 + - \cdots + (-1)^{n-1} t^{2n-2} = \frac{1 - (-1)^n t^{2n}}{1 + t^2}.$$

We can rearrange this as follows:

$$\frac{1}{1+t^2} = 1 - t^2 + t^4 - t^6 + - \cdots + (-1)^{n-1} t^{2n-2} + (-1)^n \frac{t^{2n}}{1+t^2}.$$

If we integrate this from 0 to x, we find (use the substitution $t = \tan\theta$),

$$\begin{aligned} \tan^{-1} x &= x - \frac{x^3}{3} + \frac{x^5}{5} - + \cdots + (-1)^{n-1} \frac{x^{2n-1}}{2n-1} + (-1)^n \int_0^x \frac{t^{2n}}{1+t^2}\, dt \\ &= x - \frac{x^3}{3} + \frac{x^5}{5} - + \cdots + (-1)^{n-1} \frac{x^{2n-1}}{2n-1} + (-1)^n \epsilon_n \end{aligned}$$

where

$$0 \le \epsilon_n = \int_0^x \frac{t^{2n}}{1+t^2}\, dt \le \int_0^x t^{2n}\, dt = \frac{x^{2n+1}}{2n+1}.$$

Suppose now that we fix x in the interval $0 \le x \le 1$ and let $n \to \infty$.
Then $0 \le \epsilon_n \le \frac{1}{2n+1} \to 0$, and

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - + \cdots$$

which is (3). Let us return to the game. A famous and ancient formula for π is

$$\pi = 16 \tan^{-1}\left(\frac{1}{5}\right) - 4 \tan^{-1}\left(\frac{1}{239}\right).$$

It was used some hundreds of years ago to calculate π to hundreds of decimals. Let us see if that is correct. The formulae in (2), (3) can be used to write

$$2 \tan^{-1}\left(\frac{1}{5}\right) = \tan^{-1}\left(\frac{\frac{1}{5} + \frac{1}{5}}{1 - \frac{1}{5} \times \frac{1}{5}}\right) = \tan^{-1}\left(\frac{5}{12}\right),$$

$$4 \tan^{-1}\left(\frac{1}{5}\right) = \tan^{-1}\left(\frac{\frac{5}{12} + \frac{5}{12}}{1 - \frac{5}{12} \times \frac{5}{12}}\right) = \tan^{-1}\left(\frac{120}{119}\right),$$

$$\begin{aligned} 4 \tan^{-1}\left(\frac{1}{5}\right) - \tan^{-1}\left(\frac{1}{239}\right) &= \tan^{-1}\left(\frac{120}{190}\right) - \tan^{-1}\left(\frac{1}{239}\right) \\ &= \tan^{-1}\left(\frac{\frac{120}{119} - \frac{1}{239}}{1 + \frac{120}{119} \times \frac{1}{239}}\right) = \tan^{-1}\left(\frac{120 \times 239 - 119}{119 \times 239 + 120}\right) \\ &= \tan^{-1} 1 = \frac{\pi}{4} \end{aligned}$$

and

$$\begin{aligned}\pi &= 16\tan^{-1}\left(\frac{1}{5}\right) - 4\tan^{-1}\left(\frac{1}{239}\right) \\ &= 16\left(\frac{1}{5} - \frac{1}{3\times 5^3} + \frac{1}{5\times 5^5} - + \cdots\right) \\ &\quad - 4\left(\frac{1}{239} - \frac{1}{3\times 239^3} + \frac{1}{5\times 239^5} - + \cdots\right).\end{aligned}$$

If we use just the first eight terms in the first series and the first two in the second series, we find

$$\pi \approx 3.14159265358962\ldots,$$

which is correct to 12 decimal places! Maybe you can find some interesting or useful results?

Here are some I found:

$$\begin{aligned}\frac{\pi}{4} &= \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{8}\right) \\ &= 2\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right) \\ &= 3\tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{1}{20}\right) + \tan^{-1}\left(\frac{1}{1985}\right) \\ &= \tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{6}\right) + \tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{30}\right) \\ &\qquad + \tan^{-1}\left(\frac{1}{372}\right) - \tan^{-1}\left(\frac{1}{32307}\right).\end{aligned}$$

More systematically,

$$\tan^{-1}\left(\frac{1}{n}\right) = \tan^{-1}\left(\frac{1}{n+1}\right) + \tan^{-1}\left(\frac{1}{n^2+n+1}\right), \tag{5}$$

so

$$\begin{aligned}\frac{\pi}{4} &= \tan^{-1}(1) \\ &= \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right) \\ &= \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{1}{13}\right) \\ &= \tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{13}\right)\end{aligned}$$

$$
\begin{aligned}
&= \tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{13}\right) + \tan^{-1}\left(\frac{1}{21}\right) \\
&= \tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{14}\right) + \tan^{-1}\left(\frac{1}{21}\right) \\
&\qquad + \tan^{-1}\left(\frac{1}{183}\right) \\
&= \tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{13}\right) + \tan^{-1}\left(\frac{1}{14}\right) \\
&\qquad + \tan^{-1}\left(\frac{1}{21}\right) + \tan^{-1}\left(\frac{1}{183}\right) \\
&= \text{etc.}
\end{aligned}
$$

(with all arguments different), or

$$
\begin{aligned}
\frac{\pi}{4} &= \tan^{-1}(1) \\
&= \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right) \\
&= 2\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right) \\
&= 2\tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{1}{7}\right) + 2\tan^{-1}\left(\frac{1}{13}\right) \\
&= 2\tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{7}\right) + 2\tan^{-1}\left(\frac{1}{13}\right) + 2\tan^{-1}\left(\frac{1}{21}\right) \\
&= 2\tan^{-1}\left(\frac{1}{6}\right) + \tan^{-1}\left(\frac{1}{7}\right) + 2\tan^{-1}\left(\frac{1}{13}\right) + 2\tan^{-1}\left(\frac{1}{21}\right) + 2\tan^{-1}\left(\frac{1}{31}\right) \\
&= 3\tan^{-1}\left(\frac{1}{7}\right) + 2\tan^{-1}\left(\frac{1}{13}\right) + 2\tan^{-1}\left(\frac{1}{21}\right) + 2\tan^{-1}\left(\frac{1}{31}\right) + 2\tan^{-1}\left(\frac{1}{43}\right) \\
&= 3\tan^{-1}\left(\frac{1}{8}\right) + 2\tan^{-1}\left(\frac{1}{13}\right) + 2\tan^{-1}\left(\frac{1}{21}\right) + 2\tan^{-1}\left(\frac{1}{31}\right) + 2\tan^{-1}\left(\frac{1}{43}\right) \\
&\qquad + 2\tan^{-1}\left(\frac{1}{57}\right) \\
&= \text{etc.}
\end{aligned}
$$

(using (4) on the term with the least denominator each time.) We also have

$$
\begin{aligned}
\tan^{-1}\left(\frac{1}{2n-1}\right) &= \tan^{-1}\left(\frac{1}{2n+1}\right) + \tan^{-1}\left(\frac{1}{2n^2+1}\right) \\
&\qquad + \tan^{-1}\left(\frac{1}{4n^4+2n^2+1}\right),
\end{aligned}
\tag{6}
$$

so we can keep all denominators odd.

$$
\begin{aligned}
\frac{\pi}{4} &= \tan^{-1}(1) \\
&= 2\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right) \\
&= 2\tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{7}\right) + 2\tan^{-1}\left(\frac{1}{9}\right) + 2\tan^{-1}\left(\frac{1}{73}\right) \\
&= 3\tan^{-1}\left(\frac{1}{7}\right) + 2\tan^{-1}\left(\frac{1}{9}\right) + 2\tan^{-1}\left(\frac{1}{19}\right) \\
&\qquad + 2\tan^{-1}\left(\frac{1}{73}\right) + 2\tan^{-1}\left(\frac{1}{343}\right) \\
&= \text{etc.}
\end{aligned}
$$

More importantly, perhaps,

$$
\tan^{-1}\left(\frac{1}{2n-1}\right) - \tan^{-1}\left(\frac{1}{2n+1}\right) = \tan^{-1}\left(\frac{1}{2n^2}\right). \tag{7}
$$

If we write $1,\ 2,\ \ldots,N$ for n, and add, we get

$$
\frac{\pi}{4} - \tan^{-1}\left(\frac{1}{2N+1}\right) = \sum_{n=1}^{N} \tan^{-1}\left(\frac{1}{2n^2}\right),
$$

and if we let $N \to \infty$,

$$
\frac{\pi}{4} = \sum_{n=1}^{\infty} \tan^{-1}\left(\frac{1}{2n^2}\right).
$$

This can actually be written in the following very interesting form:

$$
\begin{aligned}
\frac{\pi}{4} &= \sum_{n=1}^{\infty}\left\{\frac{1}{2n^2} - \frac{1}{3\times 2^3 n^6} + \frac{1}{5\times 2^5 n^{10}} - + \cdots\right\} \\
&= \frac{1}{2}\zeta(2) - \frac{1}{3\times 2^3}\zeta(6) + \frac{1}{5\times 2^5}\zeta(10) - + \cdots
\end{aligned}
$$

where $$\zeta(k) = \sum_{n=1}^{\infty}\frac{1}{n^k},$$

and it is known that if k is even (as it is in our identity (7)) that $\zeta(k)$ is a rational multiple of π^k ! So (7) expresses π as a series in even powers of π, which is quite exciting!

Just one closing remark: if $\{F_n\}$ denotes the Fibonacci sequence,

$$
\{F_n\}_{n\geq 0} = \{0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ \ldots\}
$$

then $$F_{2n+2}F_{2n}+1 = F_{2n+1}^2$$

and $$\tan^{-1}\left(\frac{1}{F_{2n}}\right) - \tan^{-1}\left(\frac{1}{F_{2n+2}}\right) = \tan^{-1}\left(\frac{1}{F_{2n+1}}\right).$$

If we put $n = 1,\ 2,\ \ldots\ , N$ and add, we obtain

$$\frac{\pi}{4} - \tan^{-1}\left(\frac{1}{F_{2N+2}}\right) = \sum_{n=1}^{N} \tan^{-1}\left(\frac{1}{F_{2n+1}}\right).$$

If we now let $N \to \infty$, we find

$$\begin{aligned} \frac{\pi}{4} &= \sum_{n=1}^{\infty} \tan^{-1}\left(\frac{1}{F_{2n+1}}\right) \\ &= \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{13}\right) + \ \cdots\ . \end{aligned}$$