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How to make a contact lens

Adelle Coster[1]

Recently a colleague from the Optometry school came to me with a problem. He had designed a new shaped contact lens. Unlike other lens systems this was not designed as an optical lens, but rather as a shaping lens, designed to be put in overnight. It will reshape the front of the cornea which is the transparent portion of the outer fibrous coat of the eyeball that covers the iris and the pupil.

The cornea itself is not spherical, but rather ellipsoid. It protrudes more than its width. Given that the contact lens sits at the apex of the cornea, and is round, we can simply consider the problem in the plane slicing down the middle of the eyeball.

He wanted to make a symmetrical lens comprised of several segments, the BOZR (Back Optic Zone Radius, the central segment), P1 (Peripheral Curve 1) and P2 (Peripheral Curve 2) which are elliptical, and segment T (Tangential) which is a straight section meeting the cornea at a tangential point. The specification of the lens is defined by the properties of ellipses, for the elliptical segments, and the angle at which the tangent meets the central axis of the lens for the tangential segment.

In order to work out the specifications for the lens segments, we need to build a mathematical model of the cornea-lens system. As the system is circularly symmetric around the central point of the lens, we only have to consider the model in the positive quadrant.

Figure 1 shows the model for the cornea-lens system. The particular values in which he was interested were $h_B = 0.04$mm, $h_1 = 0.008$mm, $h_2 = 0.005$mm and $h_3 = 0.02$mm, with corresponding $d_1 = 2.5$mm, $d_2 = 3.15$mm and $d_3 = 4.15$mm.

Some extra information will be needed in order to fully specify the lens. We need to know the geometry of the cornea itself, and also further specify some of the parameters of the lens segments.

The Cornea

The cornea can be considered to be an ellipse. This is centered at the origin with semi-major (x) axis a_C, semi-minor (y) axis b_C, as shown in Figure 2. Mathematically, this is described as

$$\frac{x^2}{a_C^2} + \frac{y^2}{b_C^2} = 1.$$

The ellipse has eccentricity,

$$e_C^2 = 1 - \frac{b_C^2}{a_C^2},$$

[1]Adelle Coster is an applied mathematician at the University of New South Wales.

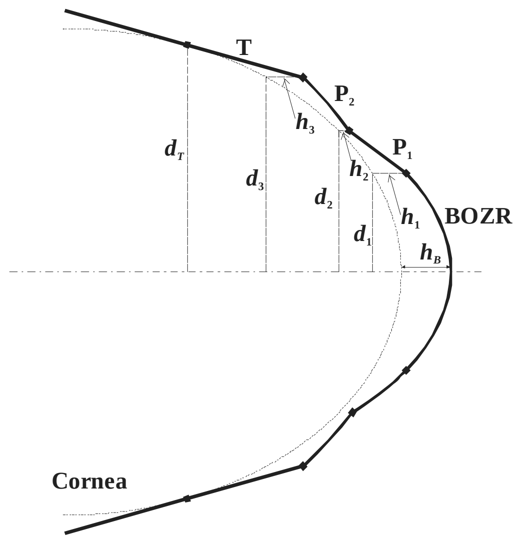

Figure 1: Model for the cornea lens system.

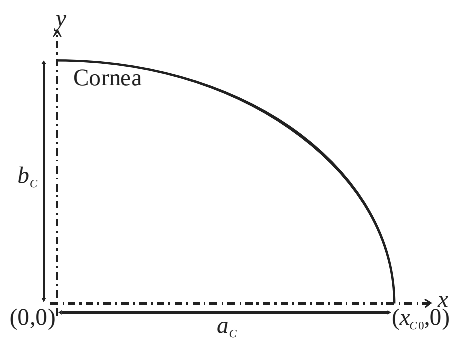

Figure 2: Portion of an ellipse with shape factor p_C.

and thus a shape factor

$$p_C = 1 - e_C^2 = \frac{b_C^2}{a_C^2}.$$

In optometry, lens designers do not use semi-major and -minor axes, but rather a quantity termed the apical radius, R_{0C}. This is defined as

$$R_{0C} = p_C a_C.$$

In order to position our lens segments, we need to be able to describe the horizontal location x on the ellipse, for any given vertical displacement y:

$$\frac{x^2}{a_C^2} + \frac{y^2}{b_C^2} = 1$$

$$x = \sqrt{a_C^2 - \frac{a_C^2 y^2}{b_C^2}}$$

$$x = \frac{1}{p_C}\sqrt{R_{0C}^2 - p_C y^2}$$

So we can completely specify the corneal points $(x_{C0}, 0)$, (x_{C1}, d_1), (x_{C2}, d_2) and (x_{C3}, d_3).

An Elliptical Lens Segment

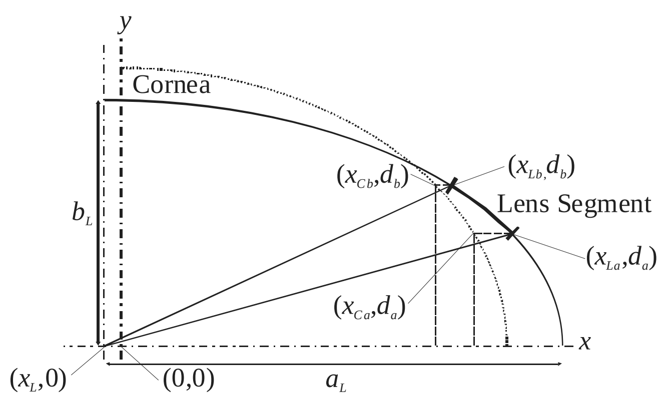

Figure 3: Typical elliptical lens segment.

Consider a typical lens segment, with endpoints (x_{La}, d_a) and (x_{Lb}, d_b), shown in Figure 3. This is located on an ellipse, centered at $(x_L, 0)$:

$$\frac{(x - x_L)^2}{a_L^2} + \frac{y^2}{b_L^2} = 1.$$

It has shape factor p_L and apical radius R_{0L}. However, we do not know both these values, but rather that the ellipse has to pass through the endpoints of the segment. So we need to be able to specify p_L and R_{0L} in terms of these.

If we rearrange the equation for the lens ellipse (as for the cornea) we have

$$x = x_L + \frac{1}{p_L}\sqrt{R_{0L}^2 - p_L y^2}.$$

We can define the x endpoints x_{La} and x_{Lb} both in terms of the lens segment ellipse and as offsets from the cornea points x_{Ca} and x_{Cb}:

$$x_{La} = x_L + \frac{1}{p_L}\sqrt{R_{0L}^2 - p_L d_a^2} = x_{Ca} + h_a \tag{1}$$

$$x_{Lb} = x_L + \frac{1}{p_L}\sqrt{R_{0L}^2 - p_L d_b^2} = x_{Cb} + h_b \tag{2}$$

Subtracting, (1) - (2),

$$\frac{1}{p_L}\sqrt{R_{0L}^2 - p_L d_a^2} - \frac{1}{p_L}\sqrt{R_{0L}^2 - p_L d_b^2} = x_{Ca} - x_{Cb} + h_a - h_b$$
$$\sqrt{R_{0L}^2 - p_L d_a^2} - \sqrt{R_{0L}^2 - p_L d_b^2} = p_L\left(x_{Ca} - x_{Cb} + h_a - h_b\right)$$

Solving this for R_{0L}, assuming p_L is known,

$$R_{0L} = \pm\frac{1}{2C}\sqrt{\left(p_L C^2 + (d_a - d_b)^2\right)\left(p_L C^2 + (d_a + d_b)^2\right)}$$

where $C = x_{Ca} - x_{Cb} + h_a - h_b$.
Given that R_{0L} is the apical radius, this will be given by the positive answer.
Conversely if you want to specify R_{0L} and calculate p_L,

$$p_L = \frac{-\left(d_a^2 + d_b^2\right) \pm 2\sqrt{d_a^2 d_b^2 + C^2 R_{0L}^2}}{C^2},$$

Again, one answer will be negative. Given that the shape factor is $p_C = \frac{b_C^2}{a_C^2}$, a negative value implies an imaginary semi-major or -minor axis. This would, instead of being an ellipse, be a hyperbola. Check for yourself to see whether the analysis works for a hyperbolic lens segment. The centre point of the ellipse is given by rearranging (1):

$$x_L = x_{Ca} + h_a - \frac{1}{p_L}\sqrt{R_{0L}^2 - p_L d_a^2}$$

given that you now know all the variables.

The Elliptical Lens Segments

You can then use the above to calculate the values for each of the lens segments.

Lens	Ellipse	Centre	Shape Factor	Apical Radius	End Points	Cornia Points
BOZR	$\frac{(x-x_B)^2}{a_B^2} + \frac{y^2}{b_B^2} = 1$	$(x_B, 0)$	$p_B = \frac{b_B^2}{a_B^2}$	R_{0B}	$(x_{B0}, 0)$, (x_{L1}, d_1)	$(x_{C0}, 0)$, (x_{C1}, d_1)
P1	$\frac{(x-x_{P1})^2}{a_{P1}^2} + \frac{y^2}{b_{P1}^2} = 1$	$(x_{P1}, 0)$	$p_{P1} = \frac{b_{P1}^2}{a_{P1}^2}$	R_{0P1}	(x_{L1}, d_1), (x_{L2}, d_2)	(x_{C1}, d_1), (x_{C2}, d_2)
P2	$\frac{(x-x_{P2})^2}{a_{P2}^2} + \frac{y^2}{b_{P2}^2} = 1$	$(x_{P2}, 0)$	$p_{P2} = \frac{b_{P2}^2}{a_{P2}^2}$	R_{0P2}	(x_{L2}, d_2), (x_{L3}, d_3)	(x_{C2}, d_2), (x_{C3}, d_3)

Tangential Lens Segment

The tangential lens segment is given by a straight line, with slope equivalent to the gradient of the cornea at the point of contact, as shown in Figure 4. It also has an endpoint joining the last elliptical lens segment at (x_{L3}, d_3). We need to be able to tell our lens manufacturer the angle θ that the tangential segment makes with the x axis, also shown in Figure 4.

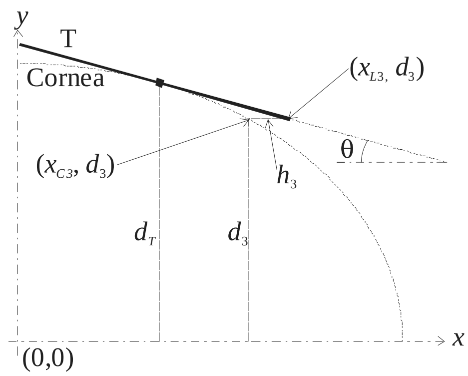

Figure 4: Typical tangential lens segment.

As we saw previously, the cornea is described by

$$\frac{x^2}{a_C^2} + \frac{y^2}{b_C^2} = 1$$

Differentiating this we have

$$\frac{2x}{a_C^2} + \frac{2y}{b_C^2}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{2x}{a_C^2}\frac{b_C^2}{2y} = -p_c\frac{x}{y}$$

If we specify that we want the point of contact to be at a y-displacement d_T, then the x-coordinate, x_T, is given by

$$x_T = \frac{1}{p_C}\sqrt{R_{0C}^2 - p_C d_T^2}$$

as it lies on the cornea ellipse.

The tangential line is $y = mx + b$. Thus the slope, m, is

$$m = \left.\frac{dy}{dx}\right|_{(x_T,d_T)} = -p_C\frac{x_T}{d_T} = -\frac{1}{d_T}\sqrt{R_{0C}^2 - p_C d_T^2}$$

We need the slope of the line between (x_L, d_3) and (x_T, d_T) to be the same as the gradient of the ellipse at (x_T, d_T).

$$\text{i.e.} \qquad \frac{d_T - d_3}{x_T - x_{L3}} = -\frac{1}{d_T}\sqrt{R_{0C}^2 - p_C d_T^2}.$$

Solving for d_T (and recalling that x_T is also dependent on d_T),

$$d_T = \frac{R_{0C}}{p_C d_3}\left(\frac{R_{0C} d_3^2 \pm x_{L3} d_3 \sqrt{p_C\left(-R_{0C}^2 + p_C\left(x_{L3}^2 p_C + d_3^2\right)\right)}}{\left(x_{L3}^2 p_C + d_3^2\right)}\right)$$

Again there are two values, only one of which is physically possible. (One may occur in the middle of one of the elliptical segments).

Once you have d_T you can then work out m, and from m the angle:

$$\tan\theta = |m| = \frac{1}{d_T}\sqrt{R_{0C}^2 - p_C d_T^2}$$

where θ is the angle between the tangent line and the x-axis. (You need the absolute value as we have a negative slope).

Finishing Up

So, can we now give our optometrist the specifications he needs to build his lens? See if you can work out the specifications for the lens given the following information. $R_{0C} = 7.668$mm, $p_C = 0.5239$, $R_{0B} = 7.198$mm, $R_{0P1} = 7.500$mm and $p_{P2} = 0.5300$, with $h_B = 0.04$mm, $h_1 = 0.008$mm, $h_2 = 0.005$mm and $h_3 = 0.02$mm, $d_1 = 2.5$mm, $d_2 = 3.15$mm and $d_3 = 4.15$mm.

What happens if you change R_{0P1} to 7.198mm? What is the shape of the lens segment? Does the theory still hold?