Solutions to Problems 1191–1200

Q1191. Let ABC be a triangle with sides a, b, c in the usual way and let r be its circumradius.

(i) Show that $3r > \dfrac{a+b+c}{2}$.

(ii) Let P be any point within ABC, and let r_1 be the circumradius of ABP. Is it true that $r_1 < r$?

ANS:

(i) Clearly $AO + OB > AB$ and so $AB < 2r$. Similarly $BC < 2r$ and $CA < 2r$. Hence

$$3r > \frac{a+b+c}{2}.$$

(ii) By taking P inside the triangle and close to the side AB, we can make r_1 as large as we please. Hence it is not true that $r_1 < r$, for every position of the point P.

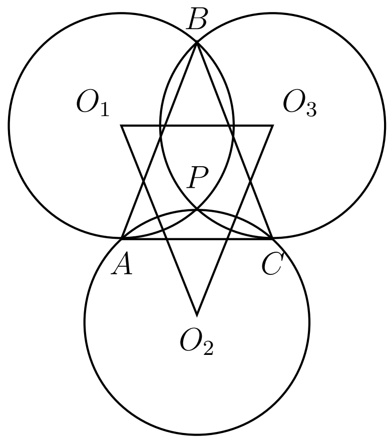
Q1192. Three circles with centres O_1, O_2, O_3 and of equal radii r, all pass through a point P. Let their second points of intersection be A, B, C. Show that O_1, O_2, O_3 is congruent to ABC. What is the circumradius of ABC? el

ANS: (submitted by Ildar Gaisin, Year 11, All Saints Anglican School, 2006).

Let A, B and C denote respectively the second point of intersection of the circles with centres O_1 and O_2, O_1 and O_3, and O_2 and O_3, as shown in the diagram. Clearly, then

$$O_1P = O_2P = O_3P = O_1A = O_1B = O_2A = O_2C = O_3B = O_3C = r.$$

Hence the quadrilaterals O_1AO_2P, O_2CO_3P and O_3BO_1P are rhombuses, implying that $O_1A \| O_3C$, $O_2C \| O_1B_1$ and $O_3B \| O_2A$. Thus the quadrilaterals O_1ACO_3, O_2CBO_1, and O_2ABO_3 are parallelograms. Hence, we must have $O_1O_3 = AC$, $O_1O_2 = BC$, and $O_2O_3 = AB$. Therefore $\Delta O_1O_2O_3$ is congruent to ΔABC, as desired. Finally, since $O_1P = O_2P = O_3P = r$, it follows that the circumradius of ΔABC is r.



Q1193. $ABCD$ is a trapezium with $AB \parallel DC$ and $AD = AB + DC$. Let M be a point on AD such that $AM = AB$.

(i) Prove that $\angle BMC$ is a right angle.

(ii) Let F be the midpoint of BC. Prove that $\angle AFD$ is also a right angle.

ANS:

(i) ABM and MDC are isosceles triangles and so $\angle AMB = \angle ABM = \alpha$ and $\angle DMC = \angle DCM = \beta$.

Since $\angle MAB + \angle MDC = 180°$, it follows that $\angle AMB + \angle DMC = 90°$.

Hence $\angle BMC = 90°$.

(ii) *This is a beautiful proof from Marianne Bruins, Hornsby Girls High School, 1997.*

Since $\angle BMC = 90°$, it follows that F is the centre of a circle through B, M, C.

Thus $FB = FM = FC$.

Now $MFCD$ is a kite and so its diagonals MC and DF are perpendicular at their point of intersection, X, say.

Similarly $ABFM$ is a kite and so its diagonals AF and BM are perpendicular at their point of intersection Y, say.

Now three of the angles of the quadrilateral $MYFX$ are right angles and so the fourth angle $XFY = \angle AFD = 90°$, as required.

Q1194. In the triangle ABC, M is the midpoint of BC. Points X on AB and Y on AC are such that $XY \parallel BC$. Show that BY and CX intersect at a point P on AM.

ANS 1:

Let P be the intersection of BY and CX and let AP meet BC at the point N.

It is our plan to prove that N is the midpoint of BC.

First the triangles BXC and BYC have the same base BC and the same height, since $XY \parallel BC$.

Hence, subtracting BPC, it follows that the triangles BXP and CYP have the same area.

Now AXY is similar to ABC, since $XY \parallel BC$.

Hence $$\frac{AX}{BX} = \frac{AY}{YC}.$$

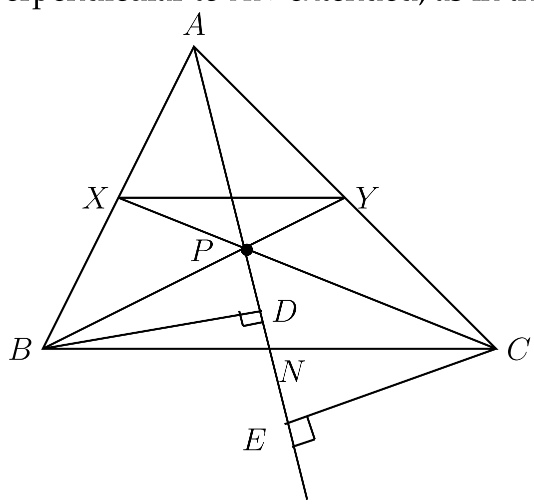
Now $$\frac{APX}{BPX} = \frac{AX}{BX} = \frac{AY}{YC} = \frac{APY}{CYP}.$$

Since BXP and CYP have the same area, it follows that APX and APY have the same area.

Hence $ABP = APX + BPX$ has the same area as $ACP = APY + CYP$.

Since these two triangles have the same base AP and the same area, they must have the same perpendicular heights.

Let BD and CE be perpendicular to AN extended, as in the diagram.



Consider the triangles BND and CNE.

First $BD = CE$, from the above argument.

Next $\angle CEN = \angle BDN = 90^\circ$ and $\angle BND = \angle CNE$.

Hence BND and CNE are congruent and so $BN = NC$, as required.

ANS 2: This answer was suggested by Ildar Gaisin, Year 11 All Anglican School, 2006.

Since $XY \| BC$ it follows that ΔAXY is similar to ΔABC. Hence

$$\frac{AC}{AB} = \frac{AY}{AX} = \frac{AC - AY}{AB - AX} = \frac{CY}{BX},$$

implying $CY \cdot AX = AY \cdot BX$. Now because M is the midpoint of BC, $BM = CM$ and thus

$$\frac{BM \cdot CY \cdot AX}{CM \cdot AY \cdot BX} = 1.$$

By Ceva's theorem AM, BY and CX are concurrent.

Editor's Note: The above solution is a beautiful application of Ceva's theorem, which can be stated as follows:

Let D, E and F be 3 points on sides BC, CA and AB of a triangle ABC. Then AD, BE and CF are concurrent (i.e. they intersect in a single point) if and only if

$$\frac{BD \cdot CE \cdot AF}{DC \cdot EA \cdot FB} = 1.$$

In fact the proof in Answer 1 is along the line of the proof of this theorem. More information on this theorem can be found in

`http//mathworld.wolfram.com/CevasTheorem.html`

Q1195. Two circles intersect at A and B and l is a variable line through A which intersects the circles at X and Y, respectively.

(i) Show that as l varies, all the triangles XBY will be similar to each other.

(ii) Find out how to draw l such that

(a) XY is as long as possible, and (b) A is the midpoint of XY.

ANS:

(i) Let X_1AY_1 be another line through A with X_1 on the first circle and Y_1 on the second circle.

Then $\angle AXB = \angle AX_1B$ on the same arc, while $\angle AYB = \angle AY_1B$ for the same reason.

Hence XYB is similar to X_1Y_1B.

(ii) (a) Since the triangles are all similar, fixing XYB for a minute,

$$X_1Y_1 = \frac{XY}{YB} Y_1B.$$

Now X_1Y_1 is a maximum when Y_1B is maximal, and this occurs when BY_1 is a diameter.
So we get the longest XY when we take X and Y such that BX and BY are diameters of their respective circles. (It is an easy exercise to see that XAY is a straight line and then XY is of maximal length.)

(b) Take any such triangle XYB and let M be the midpoint of XY.
Now join MB.
Let Y_1 be chosen on the circle such that $\angle BAY_1 = \angle BMY$.
Now extend Y_1A to X_1 as in the diagram.
Then X_1Y_1B is similar to XYB and so A is now the midpoint of X_1Y_1.

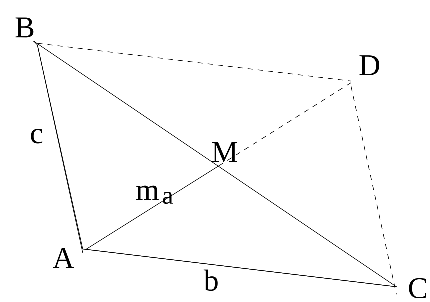
Q1196. ABC is a triangle, m_a is the median from A to the side $BC = a$.

(i) Show that $m_a < \frac{1}{2}(b + c)$.

(ii) If p is the perimeter of ABC then

$$\frac{3}{4}p < m_a + m_b + m_c < p.$$

ANS:



(i) See the figure where the line through AM has been extended to D such that $AM = MD$ and BD and DC are joined by lines.

Since M is the midpoint of BC and AD, it follows that $ABCD$ is a parallelogram and so $BD = b$.

Now $AB + BD = b + c > AD = 2m_a$.

(ii) Similarly $c + a > 2m_b$ and $a + b > 2m_c$.

Adding we get that $2p > 2(m_a + m_b + m_c)$ and this completes half the inequality.

For the other half, let X be the intersection point of the three medians of ABC.

Then
$$AX = \frac{2}{3}m_a, \quad BX = \frac{2}{3}m_b, \quad CX = \frac{2}{3}m_c.$$

Now $AX + XB = \frac{2}{3}(m_a + m_b) > AB = c$.

Similarly $\frac{2}{3}(m_b + m_c) > a$ and $\frac{2}{3}(m_c + m_a) > b$.

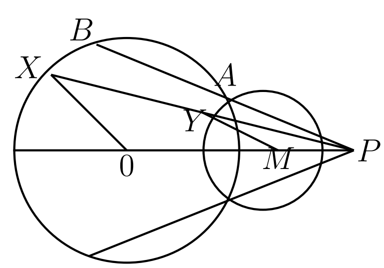
Adding we get that $\frac{4}{3}(m_a + m_b + m_c) > a + b + c = p$. This is the required result.

Q1197. Given a circle, centre 0, radius r, and a point P outside the circle, construct a line through P meeting the circle at the points A and B such that $PA = AB$.
ANS:

Let M be the midpoint of OP, and let an arbitrary line through P meet the circle O in point X. Let Y be the midpoint of PX.

Then the triangles XOP and YMP are similar. It then follows that $MY = \frac{1}{2}r$.

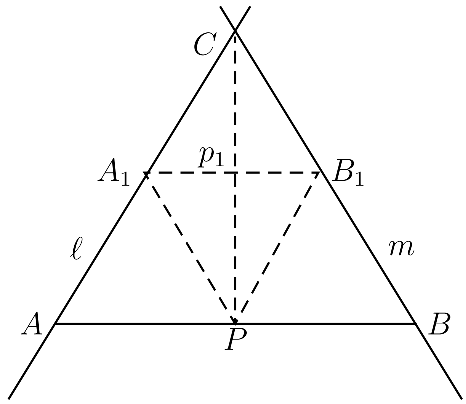
If we let X traverse the circle with origin O then the midpoint of the line XP at Y will describe a circle with origin at M with radius $= \frac{1}{2}r$. We may consider this circle as having been obtained from O by a similarity transformation, centre P. Circle M and circle O intersect at point A and PA extended meets the circle O at B. (As it is clear from the figure, there are two solutions.)



Q1198. Given two intersecting lines ℓ and m and a point P not lying on either line, construct a straight line through P meeting ℓ in A and m in B such that P is the midpoint of AB.

ANS:

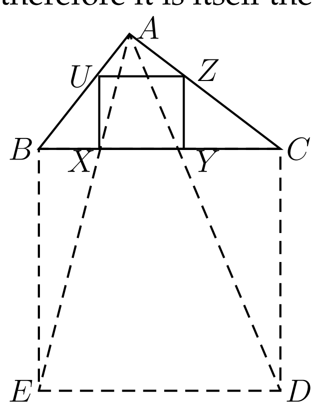
Let ℓ intersect m at C. Draw a line through P parallel to m, meeting ℓ at A_1. Similarly, let PB_1, parallel to ℓ, meet m at B_1. Let the diagonals of the parallelogram CA_1PB_1 meet at P_1. Then P_1 is the midpoint of A_1B_1 and any line segment parallel to A_1B_1, cuts ℓ and m at two points such that their midpoint is on the line CP. So the answer is draw a line through P parallel to A_1B_1.



Q1199. Given a $\triangle ABC$, construct a square $XYZU$ such that side XY lies along BC, while vertex Z is on AC, vertex U on AB.

ANS:

We construct a square $BCDE$, external to the triangle, on the side BC. Join D and E to the opposite vertex A, meeting BC at Y and X respectively. Erect perpendiculars to BC at X and Y, meeting AB at U, AC at Z. The quadrilateral $UZYX$ is centrally similar to the square $BCDE$, therefore it is itself the required square.



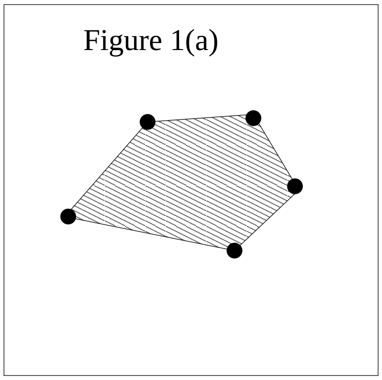
Q1200. The 'Happy Ending' Problem.

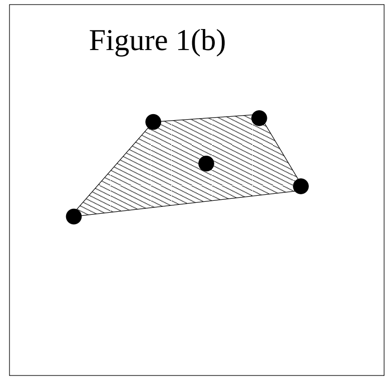
Given any five points in the plane with no three points lying on a straight line show that it is always possible to select four of the points as vertices for a convex quadrilateral. (Note that a quadrilateral is convex if any two points inside the quadrilateral can be connected by a straight line segment that does not fall outside the quadrilateral.)

ANS: The answer provided below is from the article "The 'Happy-End' Problem" by D. Harvey in *Parabola* Vol 38 No 1 (2002) 5–9.

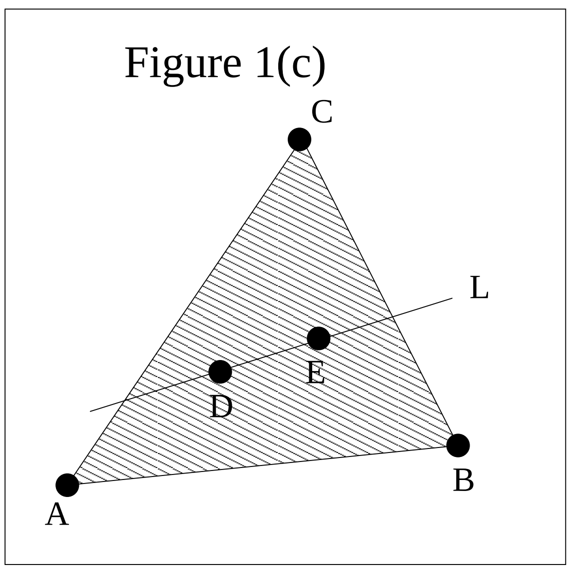
Esther's elegant proof of this fact proceeds as follows. First we have to construct the 'convex hull' of the five given points. The easiest way to visualise this is to imagine that the points represent nails sticking out of a wooden board, and that we stretch an elastic band around the nails. The convex hull is the polygon formed by the elastic band.

In Figure 1(a), the boundary of the convex hull includes all five points, so it is a pentagon. In this case, we can choose any four points we like, and they will form a convex quadrilateral.





In Figure 1(b), the boundary of the convex hull only includes four of the points, with the last remaining point strictly inside the convex hull. In this case, we obviously choose the four points on the outside to be our convex quadrilateral.



In Figure 1(c), we have the trickiest situation. Here the convex hull only includes three of the points (say A, B, and C), with the remaining two points (say D and E) strictly inside the convex hull. Let L be the line joining D and E. Since the five points are in general position, the line L cannot include any of A, B or C. It is fairly easy to see that two of the external points (in this case, A and B) must lie on one side of L, and the other point (in this case C) must lie on the other. Therefore, the two interior points, and the two points which lie on the same side of L, together form a convex quadrilateral, as shown in Figure 1(c). This completes the proof!