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UNSW School Mathematics Competition 2006

Problems and Solutions

Junior Division

Problem 1. An American football field is 100 yards long, and its width is half the average of its length and its diagonal. Find its area.

Solution: $5333\frac{1}{3}$ square yards.

Suppose the width of the field is x yards. Then

$$x = \frac{1}{2} \cdot \frac{100 + \sqrt{100^2 + x^2}}{2},$$

$$(4x - 100)^2 = 100^2 + x^2,$$

$$x = 53\frac{1}{3}.$$

Problem 2. We want to tile a rectangle with dominoes. A domino is a 2×1 rectangle. A 2×2 rectangle can be tiled in 2 ways — the two dominoes may be horizontal, or they may be vertical.

(a) In how many ways can you tile a 2×3 rectangle with dominoes?

(b) In how many ways can you tile a 2×5 rectangle with the right–hand–most domino vertical?

(c) In how many ways can you tile a 2×5 rectangle with the two dominoes at the right-hand end horizontal?

(d) In how many ways can you tile a 2×10 rectangle?

(e) Answer the same questions if all the dominoes are made up of a black square and a white square. (Now there are eight ways of tiling a 2×2 square.)

Solution:

(a) 3 (all vertical, or the first two horizontal, then one vertical, or the first vertical, then two horizontal).

(b) 5 (all vertical, or the first two horizontal, the next two vertical, or the first vertical, then two horizontal, then one vertical, or the first two vertical, then the next two horizontal, or two horizontal, then two more horizontal).

(c) 3.

(d) 89.

The relevant sequence is $\{1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ \cdots\}$. These are "shifted" Fibonacci numbers, given by the recurrence $f_n = f_{n-1} + f_{n-2}$, $f_1 = 1$, $f_2 = 2$.

(e) (a) 24 (b) 160 (c) 96 (d) 91136

The sequence is

$\{2,\ 8,\ 24,\ 80,\ 256,\ 832,\ 2688,\ 8704,\ 28160,\ 91136,\ \cdots\}$.

The relevant recurrence is $g_n = 2g_{n-1} + 4g_{n-2}$. The solution is $g_n = 2^n f_n$.

Problem 3. Find consecutive positive integers, a, b, c, d and e, such that $a+b+c+d+e$ is the square of an integer, and $b+c+d$ is the cube of an integer.

Solution: $1123,\ 1124,\ 1125,\ 1126,\ 1127.$

Let the central number be n. Then

$$5n = x^2,\ \ 3n = y^3.$$

It follows that $3x^2 = 5y^3$, $x = 5a$, $y = 3b$, $3 \times 25a^2 = 5 \times 9b^3$, $5a^2 = 3b^3$, $a = 3c$, $b = 5d$, $5 \times 9c^2 = 3 \times 125d^3$, $c^2 = 25d^3$, $d = p^2$, $c = 5p^3$, $a = 15p^3$, $b = 5p^2$, $x = 75p^3$, $y = 15p^2$ and $n = 1125p^6$.

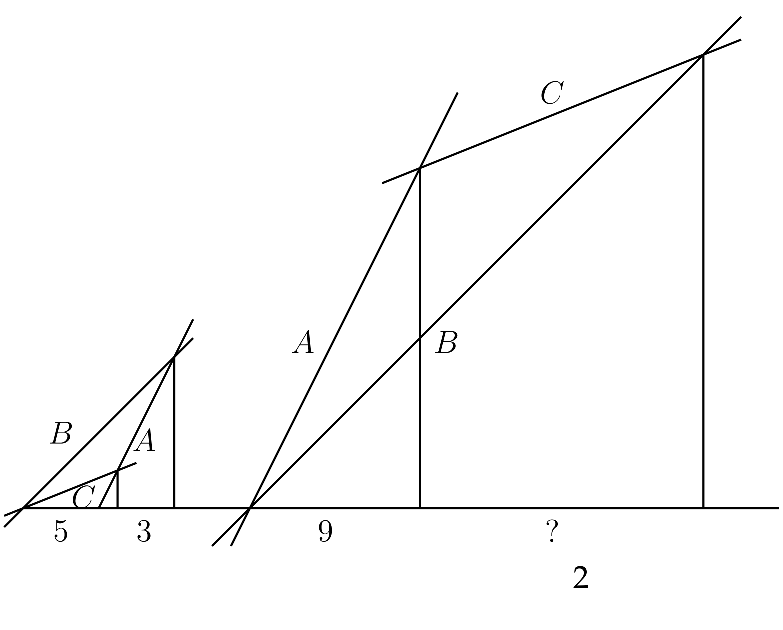
The smallest solution is $p = 1$, $n = 1125$.

Problem 4. Three motorists, A, B and C, often travel on the motorway, and each always travels at a constant speed, A the fastest, C the slowest.

One day when the three are travelling in the same direction, B overtakes C, five minutes later A overtakes C, and in another three minutes A overtakes B.

On another occasion when they are all travelling in the same direction, A overtakes B first, then nine minutes later, A overtakes C. How much later does B overtake C?

Solution: 15 minutes. Consider the following graphs:



The two triangles are similar. The result follows.

Problem 5. Two circular discs, of radii r and $r\sqrt{3}$ respectively, are placed on a plane in such a way that their edges cross at right angles. Find the area of their overlap (that is, intersection), and the area covered by their union.

Solution: Area of overlap

$$= \frac{1}{6} \times 3\pi r^2 + \frac{1}{3} \times \pi r^2 - 2 \times \frac{1}{2} \times r\sqrt{3} \times r = \left(\frac{5}{6}\pi - \sqrt{3}\right) r^2.$$

Area covered by union

$$= \pi r^2 + 3\pi r^2 - \left(\frac{5}{6}\pi - \sqrt{3}\right) r^2 = \left(\frac{19}{6}\pi + \sqrt{3}\right) r^2.$$

(The distance between centres is $2r$.)

Problem 6. Four boys have first names Alexander, Barry, Charles and David, and have the same set of names as their family names.

No boy has the same first name and family name. The family name of Charles is not Alexander. The family name of the boy whose first name is Barry is the same as the first name of the boy whose family name is the same as the first name of the boy whose family name is David.

What are the full names of the four boys?

Solution: The boys are Barry Alexander, David Barry, Alexander Charles and Charles David.

Using initials, we have the following names:

$$BX,\ XY,\ YD, \text{ where } X \neq B,\ Y \neq D,\ X \neq Y.$$

We find the following possibilities:

$$\begin{array}{ll}
X = A,\ Y = B,\ \ BA,\ AB,\ BD & no! \\
X = A,\ Y = C,\ \ BA,\ AC,\ CD,\ DB & yes! \\
X = C,\ Y = A,\ \ BC,\ CA,\ AD & no! \\
X = C,\ Y = B,\ \ BC,\ CB,\ BD & no! \\
X = D,\ Y = A,\ \ BD,\ DA,\ AD & no! \\
X = D,\ Y = B,\ \ BD,\ DB,\ BD,\ AC,\ CA & no! \\
X = D,\ Y = C,\ \ BD,\ DC,\ CD & no!
\end{array}$$

Senior Division

Problem 1. We want to tile a rectangle with dominoes. A domino is a 2×1 rectangle. A 2×2 rectangle can be tiled in 2 ways — the two dominoes may be horizontal, or they may be vertical.

(a) In how many ways can you tile a 2×3 rectangle with dominoes?

(b) In how many ways can you tile a 2×5 rectangle with the right–hand–most domino vertical?

(c) In how many ways can you tile a 2×5 rectangle with the two dominoes at the right–hand end horizontal?

(d) In how many ways can you tile a 2×10 rectangle?

(e) In how many ways can you tile a $2 \times n$ rectangle?

(f) Now there are white/white dominoes and black/white dominoes. Answer the same questions.

Solution: For parts (a), (b), (c) and (d) see Junior Problem 2.

(e) f_n, the "shifted" Fibonacci number, given by $f_n = f_{n-1} + f_{n-2}$, $f_1 = 1$, $f_2 = 2$.

(f) (a) 81 (b) 1215 (c) 729 (d) 5255361 (e) $3^n f_n$.

The recurrence here is $h_n = 3h_{n-1} + 9h_{n-2}$, with $h_1 = 3$, $h_2 = 18$. The solution is $h_n = 3^n f_n$.

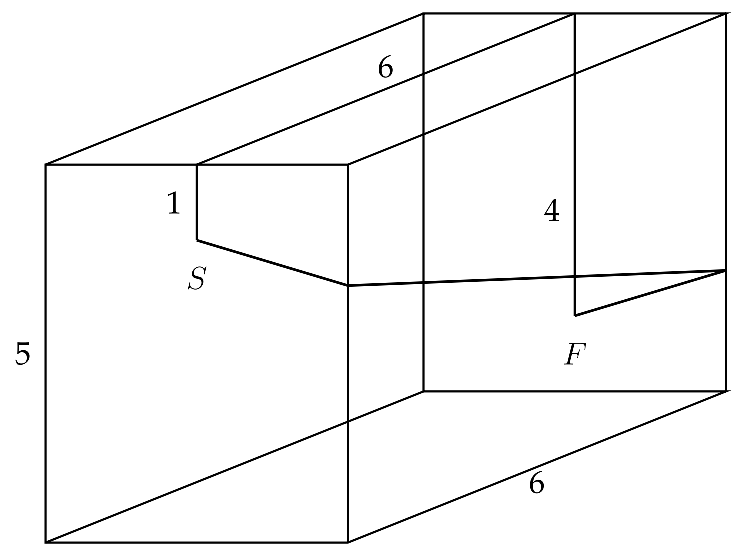
Problem 2. A spider is sitting on the end wall of a room that is $4m$ wide, $6m$ long and $5m$ high. He is $1m$ from the ceiling, and $2m$ from each of the side walls. He spies his dinner, a clever but lazy fly, sitting at the other end of the room, $1m$ from the floor and $2m$ from each side wall.

The spider could simply run up to the ceiling, along the ceiling and down the opposite wall to catch his dinner, but he is a clever fellow, and realises that he can reach his target by a shorter route! Can you find the length of his shortest route to the fly?

The fly, on the other hand, is not only clever enough to see how the spider can get to her by the shortest possible route, but also realises that if she walks a little way towards the floor, she can get further away from the spider. How far should she walk in order to get as far as possible away from the spider, and now how far is she from the spider by his shortest possible route?

Solution:

(a) By cutting down and across to the side wall, along and down the long side wall and then across and down the far wall, as shown in the diagram, the shortest route originally had length $\sqrt{(2+6+2)^2 + (5-1-1)^2} = \sqrt{109} < 11 = 1+6+4$.



(b) The fly should walk down the far wall by x m, where x satisfies the equation

$$\sqrt{10^2 + (3+x)^2} = 4 + 6 + (1 - x) = 11 - x,$$

$$109 + 6x + x^2 = 121 - 22x + x^2,$$

$$28x = 12.$$

So $x = \frac{3}{7}$, and the shortest route is now $11 - \frac{3}{7} = 10\frac{4}{7} > \sqrt{109}$.

Problem 3.

(a) Show that the numbers $\{1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7\}$ are the only seven positive integers with sum 28 and product 5040.

(b) Show that $\{1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8\}$ are **not** the only eight positive integers with sum 36 and product 40320.

Solution:

(a) Since the product of the seven numbers is 5040, one of the numbers, x say, is a multiple of 7, and since the seven numbers total 28, $x = 21,\ 14$ or 7.

If $x = 21$, the other six numbers total 7, so their product $= 1 \times 1 \times 1 \times 1 \times 1 \times 2 = 2 < 5050/21$. So $x \neq 21$.

If $x = 14$, the other six numbers total 14, and their product $\leq 2 \times 2 \times 2 \times 2 \times 3 \times 3 = 144 < 5040/14$. So $x \neq 14$.

So $x = 7$, and the other six numbers total 21, and their product is 720.

One of the six numbers, y say, is a multiple of 5, and since the six numbers total 21, $y = 15,\ 10$ or 5.

If $y = 15$, the other five numbers total 6, and their product $= 1 \times 1 \times 1 \times 1 \times 2 = 2 < 720/15$. So $y \neq 15$.

If $y = 10$, the other five numbers total 11, and their product $\leq 2 \times 2 \times 2 \times 2 \times 3 = 48 < 720/10$. So $y \neq 10$.

So $y = 5$, and the remaining five numbers total 16, and their product $= 5040/(7 \times 5) = 144 = 2^4 \times 3^2$.

At least one of the five numbers , z say, is a multiple of 3, and since the five numbers total 16, $z = 12,\ 9,\ 6$ or 3.

If $z = 9$, the other four numbers total 7, and their product $\leq 1 \times 2 \times 2 \times 2 = 8 < 144/9$. So $z \neq 9$.

We now know that the five numbers include **two** multiples of 3, and that each of the five is either a power of 2 or $3\times$a power of 2.

There are just four possibilities:

$$\begin{aligned}
&6 \times 6 \times 1 \times 1 \times 2 = 72,\\
&6 \times 3 \times 1 \times 2 \times 4 = 144,\\
&3 \times 3 \times 1 \times 1 \times 8 = 72,\\
&3 \times 3 \times 2 \times 4 \times 4 = 288.
\end{aligned}$$

Only the second of these gives a product of 144. So the seven numbers are $\{1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7\}$.

(b) There are four sets of eight numbers with sum 36 and product 40320. They are

$$\{1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8\},\ \ \{1,\ 2,\ 4,\ 4,\ 4,\ 5,\ 7,\ 9\},$$

$$\{1,\ 3,\ 3,\ 4,\ 4,\ 4,\ 7,\ 10\},\ \ \{2,\ 2,\ 2,\ 3,\ 4,\ 6,\ 7,\ 10\}.$$

Problem 4. In the New Monian mens' cricket eleven:

> all the married players are either professional and over 30, or are under 30 with an annual income of \$100000;
>
> the unmarried players are amateurs earning \$100000 per annum, or are under 30 with an income of \$60000 per annum;
>
> there is no player on \$100000 per annum who is not either married and amateur, or professional and over 30;
>
> if a player is professional and over 30, or if he is amateur and under 30, then either he is a married man on \$60000 per annum or a single man on \$100000 per annum.

How many amateurs are there on the team?
What is the total annual income of the team members?

Solution: There are no amateurs on the team, and the total annual income of the team is \$660000.

A player can belong to M, the set of married men, or to $\overline{M}$, the set of unmarried (single) men; he can belong to P, the set of professional players, or to $\overline{P}$, the set of amateurs; he can belong to Y, the set of (young) people under 30, or to $\overline{Y}$, the set of people over 30 (apparently 30-year–olds belong to one of Y or $\overline{Y}$, but this is irrelevant here); a player can belong to W, the well–paid people (on an annual salary of \$100000), or to $\overline{W}$, those on \$60000 per annum.

There are sixteen classes of people:

$$
\begin{aligned}
&(1)\ M \cap P \cap Y \cap W,\\
&(2)\ M \cap P \cap Y \cap \overline{W},\\
&(3)\ M \cap P \cap \overline{Y} \cap W,\\
&(4)\ M \cap \overline{P} \cap Y \cap W,\\
&(5)\ M \cap P \cap \overline{Y} \cap \overline{W},\\
&(6)\ M \cap \overline{P} \cap Y \cap \overline{W},\\
&(7)\ M \cap \overline{P} \cap \overline{Y} \cap W,\\
&(8)\ M \cap \overline{P} \cap \overline{Y} \cap \overline{W},\\
&(9)\ \overline{M} \cap P \cap Y \cap W,\\
&(10)\ \overline{M} \cap P \cap Y \cap \overline{W},\\
&(11)\ \overline{M} \cap P \cap \overline{Y} \cap W,\\
&(12)\ \overline{M} \cap \overline{P} \cap Y \cap W,\\
&(13)\ \overline{M} \cap P \cap \overline{Y} \cap \overline{W},\\
&(14)\ \overline{M} \cap \overline{P} \cap Y \cap \overline{W},\\
&(15)\ \overline{M} \cap \overline{P} \cap \overline{Y} \cap W,\\
&(16)\ \overline{M} \cap \overline{P} \cap \overline{Y} \cap \overline{W}.
\end{aligned}
$$

The first two conditions mean that there are no players in classes (2), (6), (7), (8), (9), (11), (13) or (16), leaving only classes (1), (3), (4), (5), (10), (12), (14) and (15). The third condition eliminates classes (1), (12) and (15), so we are left with classes (3), (4), (5), (10) and (14). The fourth condition rules out classes (3), (4) and (14), leaving only classes (4) and (10). All members of classes (4) and (10) are professionals on \$60000 per annum.

Problem 5. Find all the solutions in positive integers of the diophantine equation

$$\frac{x+y}{x^2-xy+y^2}=\frac{10}{111}.$$

Solution: $\{x,\ y\}=\{9,\ 21\}$.

Let $x+y=s,\ \ xy=p$. Then

$$\frac{s}{s^2-3p}=\frac{10}{111},$$

or,

$$30p=10s^2-111s.$$

It follows that $10|s$ and $3|s$, so $30|s$.

Also, the AM–GM inequality, or the fact that $(x-y)^2\geq 0$, gives $4p\leq s^2$. It follows that $60p=20s^2-222s\leq 15s^2$, $5s^2\leq 222s$, $s\leq 44$.

So we find $s=30$, $p=189$, $x=21$, $y=9$ (or *vice versa*).

Problem 6. Four ants are situated at the corners of a square. Each one faces the next one anticlockwise around the square. They all start moving at the same time, and continue walking directly towards the next ant. Clearly, their paths spiral in to the centre of the square. How far does each ant move along its path?

Solution: Each ant's path length is equal to the side of the square.

It is easier to solve this problem if we suppose there are n ants, at the vertices of an n–sided polygon.

Consider one of the ants, and suppose that at some point in time he is r units from the centre of the polygon. Draw the ray from the centre of the polygon to the ant. The ant is heading off at an angle of $90^\circ-\dfrac{180^\circ}{n}$ to this ray. Suppose he moves a small distance dl in this direction. Then his distance from the centre diminishes by a small amount, dr, where

$$dr=dl\cos\left(90^\circ-\frac{180^\circ}{n}\right)=dl\sin\left(\frac{180^\circ}{n}\right),$$

so

$$dl=\frac{dr}{\sin\left(\dfrac{180^\circ}{n}\right)}.$$

It follows that, while r is diminishing from its initial value r_0, say, to 0, the length of the path is increasing (from 0) to

$$l=\frac{r_0}{\sin\left(\dfrac{180^\circ}{n}\right)}.$$

On the other hand, the side of the polygon, s, satisfies

$$s = 2r_0 \sin\left(\frac{180^{\text{o}}}{n}\right).$$

So

$$l = \frac{s}{2\sin^2\left(\frac{180^{\text{o}}}{n}\right)}.$$

In the case $n = 4$, $l = s$.