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The Curved Universe

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Introduction

What shape is the Universe we live in? Does it go on for ever (like an infinite piece of paper) or if we go far enough will we end up back where we started (like on a sphere)?

These questions are fundamental to the science of **cosmology**. In order to answer them we need a theory about how matter and energy, the stuff of the Universe act over the immense distances of space and furthest reaches of time. The best theory we have for this is Einstein's General Theory of Relativity, which is theory of Gravitation. In General Relativity, gravity is how we perceive the curvature of space and time.

To explain what this means we need to get some way of defining just what 'curvature' is. To do this we go down from our four-dimensional spacetime and begin with one-dimensional curves.

Curvature of Curves

Suppose we have a piece of infinitely stiff wire as a model of a curve — infinitely stiff means the wire has come to us bent, but we cannot bend it ourselves. We need to ask ourselves how we would measure how curved the wire is, given that all we have is the wire. If you think about it you might realise that the key is how fast the direction of the wire — or curve — is changing. If the wire has a sharp curve in it, then the direction of the curve is changing very quickly; if it is nearly straight the direction is not changing much. Of course, we also realise that this works for a line — the direction of a line does not change at all, so it will have curvature zero.

In order to actually calculate curvature, we need to describe the curve in some way. So imagine the curve in a plane with the usual x and y axes. Pick a point on the curve to call the start and let s be the distance along the curve from the start point. We call s the **arc length**. Then the points on the curve are given by two functions of s, which we may as well call x and y.

At each point of the curve the direction of the curve is given by the angle of its tangent. We can show that if $x' = \dfrac{dx}{ds}$ is not zero, then we can think of the curve as the graph of a function, and its slope, $\dfrac{dy}{dx}$, will be $\dfrac{y'}{x'}$. Now the derivative tells us the slope of the tangent line, and the angle θ that the tangent line makes with the x-axis has the

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derivative as its tangent, that is

$$\tan\theta = \frac{y'}{x'}.$$

Differentiating this with respect to s and using the chain rule on the left and the quotient rule on the right gives us

$$(\sec^2\theta)\,\theta' = \frac{y''x' - x''y'}{(x')^2}.$$

But $\sec^2\theta = 1 + \tan^2\theta = 1 + \left(\frac{y'}{x'}\right)^2$. So using the symbol κ (the Greek letter 'kappa') for curvature we find that

$$\kappa(s) = \frac{d\theta}{ds} = \frac{y''x' - x''y'}{(x')^2 + (y')^2}.$$

It turns out that the denominator in this formula is always 1, but I'll not prove that here.

For example, consider the circle of radius R with centre at the origin. If we use radians to measure the angle at the centre of the circle, then an arc of length s subtends angle s/R. So if we start measuring the arc length form the point $(R, 0)$, we can describe the circle by $x(s) = R\cos(s/R)$, $y(s) = R\sin(s/R)$. When we calculate the curvature we get

$$\begin{aligned}\kappa &= \frac{\left(-(\frac{1}{R})\sin(\frac{s}{R})\right)\left(-\sin(\frac{s}{R})\right) - \left(-\frac{1}{R}\cos(\frac{s}{R})\right)\cos(\frac{s}{R})}{\left(-\sin(\frac{s}{R})\right)^2 + \left(\cos(\frac{s}{R})\right)^2} \\ &= \frac{1}{R}\frac{\sin^2(\frac{s}{R}) + \cos^2(\frac{s}{R})}{\sin^2(\frac{s}{R}) + \cos^2(\frac{s}{R})} = \frac{1}{R}.\end{aligned}$$

So a circle has constant curvature — this seems reasonable, as a circle looks equally curved all the way around. Also, we see that the smaller the radius, the larger the curvature: a small circle curves more than a large one.

It is not obvious from this work, but we can prove that we get the same value of κ wherever we put the curve in the plane. In other words, κ is an **intrinsic** property of the curve, that is it depends on the curve itself and not how we describe it. Even more importantly, κ gives a complete description of the curve: if we had a hypothetical ant constrained to live on the curve, this one function tells her everything about her path. In figure 1 we have part of the curve (there is only one) whose curvature is simply s.

These two properties of κ are just the sort of thing we will need when it comes to the curvature of the Universe, since all our measurements are internal to the Universe.

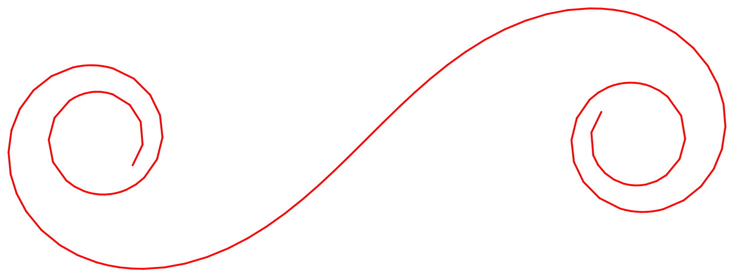

Figure 1: $\kappa(s) = s$

Curvature of Surfaces

Curves are one-dimensional. Given a curve we only need one number to describe the points on it (we used arc length s in the analysis of the previous section). Surfaces are of course two-dimensional. We need two numbers to describe the points on a surface — on the surface of the Earth we often use latitude and longitude, for example. But this means there are infinitely many different directions on a surface. If we want to define curvature of a surface using the idea of 'rate of change of direction' then we need to take into account all these directions.

One of the greatest achievements of possibly the greatest mathematician of all, Carl Freidrich Gauss, was to prove in 1828 that there is a sensible way of doing this. Take a point, P, on a surface and picture the normal line, which is perpendicular to the surface at P. A plane that goes through P and contains the normal line is called a **normal plane** at P. The intersection of a normal plane and the surface is a plane curve though P. As you rotate the normal planes around the normal line, the curve will change and so may its curvature at P.

Leonhard Euler proved in 1760 that either every such curve will have the same curvature at P or there will be a minimum and maximum value to the curvatures at P — in fact the minimum and maximum curves will be right angles. Euler called these two extremes of curvature **principal curvatures**; in the constant curvature case, we say the principal curvatures are equal.

Principal curvatures give you a qualitative idea of the shape of the surface at P, see Figure 2. If they are both the same sign, the surface must look roughly spherical; if they are opposite signs the surface is saddle shaped (like a Pringles chip); if one is zero the surface looks like a cylinder; if both are zero we have a plane.

Now given two principal curvatures, how should we get one curvature for the point P? Well, there are two obvious ways of doing this:

Firstly we could take their average, and get what is called the **mean curvature**, denoted H. Mean curvature has its uses. For example, take a closed loop (which might not be plane) and imagine a surface with that loop as edge. Then of all such surfaces, the one with smallest area has mean curvature zero.

The downside of mean curvature is that you can change it by bending the surface. Take a flat piece of paper (which has principal curvatures zero, and so $H = 0$) and

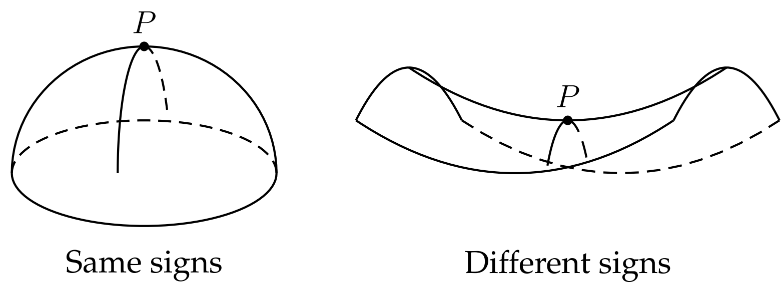

Figure 2: Curvatures

bend it into a cylinder. The cylinder has one principal curvature zero (the straight line along the cylinder) and one non-zero, as any circular cross section has non-zero κ. So the cylinder has $H \neq 0$. We see that H is not intrinsic to the surface, since to calculate it we need to know how the surface sits in space, and you need to leave the surface to measure this.

The other obvious way of combining the principal curvatures is to multiply them, and this gives us the **Gaussian curvature**, denoted K. We easily see that the Gaussian curvature does not change when we bend the plane into the cylinder, since we still have a zero principal curvature after the bending.

Gauss' great result, which he called his *Theorema Egregium* ('remarkable theorem') is that this is true for every surface: Gaussian curvature is always unchanged under bending. This means that, like κ for a curve, K is intrinsic: it is a property of the surface, not of how we describe it. We can also prove that if there were strange two-dimensional creatures that lived on a surface, there is one function, K, (of two variables) that tells them all there is to know about the world they live in, at least over small parts of it.

As an illustration of the power of the *Theorema Egregium*, we can show that maps of the earth must be distorted, something suspected by cartographers for years before Gauss proved it. Now we can only make a distortion-free map if we can bend the plane into a sphere. From what I said above, the plane has $K = 0$. So if we can prove that the sphere has non-zero K, then we are done. But this is easy: if we intersect the sphere of radius R with a normal plane then the curve of intersection is always a circle of radius R. So the principal curvatures of the sphere are all $\frac{1}{R}$, and a sphere of radius R has $K = \frac{1}{R^2}$. As this is not zero, we have our result: maps of the earth are always distorted.

Higher Dimensions

The Universe we live in has four dimensions, three space and one time. We can define higher dimensional analogues of surfaces called **manifolds** to cover this situation, and can then define what it means for these things to be curved. Einstein's General Relativity tells us that spacetime is a four-dimensional manifold and the effect of matter and energy is to curve this manifold and we feel the curvature as gravity.

Once we get past two dimensions, though, we usually need more that one number to define curvature, and so the mathematics becomes much more complicated. For four-dimensional spacetime, for example, we find we need 20 functions (each depending on four variables). For our universe, the so-called Standard Model begins by assuming the universe is, on large scales, very simple. In particular we assume that any one point of space is the same as any other point, and every direction is the same as any other. In two dimensions, a sphere has both these properties too, but a cylinder does not — every point on a cylinder is the same as any other, but not all directions are equivalent.

The advantage of our assumptions is that the usual 20 functions of four variables for the curvature reduce to one function of one variable (time). We call this function R, and time t. Then R satisfies a relatively simple differential equation called **Friedmann's Equation**:

$$\left(\frac{dR}{dt}\right)^2 = \frac{C}{R} + \frac{1}{3}\Lambda R^2 - k$$

Here the constant C is related to the density of matter and energy in the universe and Λ is a constant from the theory of General Relativity (and might be zero). The constant k tells us the shape of space. If space is flat then $k = 0$; if it has positive curvature (like the sphere) then $k = 1$; if space is more like the saddle shape we saw above $k = -1$.

Cosmologists make measurements of the amount of matter in the Universe, and how the Universe is behaving over time and use this and Friedmann's equation to work out what k is, and that is how we find out the shape of the Universe.