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Time, Length and Relativity

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Introduction

This article is a combination of two articles I wrote over 12 years ago, the first 'Time and Relativity' (sections Special Relativity to The Twin Paradox) appeared in *Parabola* in 1993 and the second 'Length and Relativity' (sections Measuring Length to The Magician's Assistant Paradox) in 1994. I have put these two articles together for this special edition of Parabola, and added some updated references.

Special Relativity

Common sense tells us that either two things happen at the same time or they do not. But a hyper-accurate atomic clock that is taken on a fast train moving at a constant velocity will run slowly when compared to an identical clock left at the station. If common sense is correct, which clock do you believe — or do you believe that physics is different in a moving train?

Albert Einstein's Special Theory of Relativity (first proposed in 1905) leads us to the, at first startling, conclusion that for a moving object time (and length) will change when measured by a second observer. Moving clocks run slowly; a moving ruler gets shorter; the order that distant events occur can be different. Experiments like the one suggested above have been done and show that the effects of Special Relativity are measurable: clocks taken on air trips run slowly. In this article we will look at some of the at first sight strange and mysterious effects that Special Relativity predicts.

Special Relativity begins with two postulates relating to the world of our experience and how we measure it. The first postulate seems harmless enough, and was used in a restricted form by Newton: *the laws of Mechanics are the same to any inertial observer*. Recall that an *inertial observer* is one for whom Newton's First Law holds: *an object on which no force acts moves in a straight line at constant speed*. For example, if you are going around on a roundabout then you are not an inertial observer as if you roll a ball away from you it follows a curved path as you measure it on your roundabout. Einstein extended this postulate to cover all physical processes, which by then included electromagnetism and optics (which by the work of James Clerk Maxwell had become a branch of electromagnetism).

The second postulate of Special Relativity also seems innocuous, and in fact when Einstein proposed Special Relativity there was some experimental evidence for it: *there is an inertial observer for whom light signals in a vacuum travel at a constant speed in all directions whatever the motion of the light source*. This constant speed, usually called c, is about $2.988 \times 10^8 ms^{-1}$. If this postulate were not true then the motion of an object moving backwards and forwards would appear jerky as its speed keeps changing.

Combining these two postulates leads us to the apparently absurd conclusion that, in vacuum, light travels at speed c in all directions at all times according to all inertial observers (however fast they are going). This is very different to our everyday experience. If an apple core is thrown out of a moving car then the core's velocity according to a kangaroo travelling in a paddock at the side of the road is the sum of the car's velocity, the velocity with which the core was thrown out of the window and the 'roo's own velocity. Light is not like that: however fast you move your torch and however fast the person seeing it is going, the light it produces travels at the same speed.

Measuring Time

We can get a grip on how the speed of an object affects the rate of a clock using some very simple mathematics. We simplify by using the first postulate and deal with the case of only one space dimension. We draw diagrams as in Figure 1, where the vertical represents time and the horizontal the one space dimension. We will perform all our measurements with light signals, as light moves at constant velocity. We also scale the axes on these diagrams so that light signals move on lines at $45°$. So anyone or anything travelling at less than the speed of light will follow a line at an angle of less than $45°$ to the vertical.

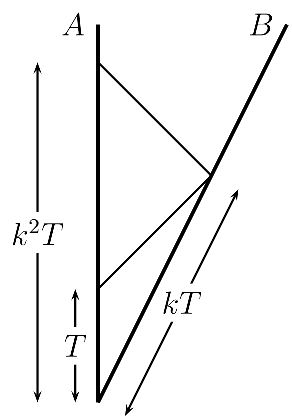

Figure 1: Two Observers

In Figure 1 we have two observers, Alan and Beryl with Beryl travelling at constant speed relative to Alan. They have identical clocks, which they sychronise to zero as they pass each other, and of course each measures time with their own clock. After time T observer Alan sends out a light signal. This reaches Beryl when her clock reads kT, say, for some real number k which must be greater than 1, as the light has had to travel some distance. Beryl immediately sends out a light signal back to Alan which must reach Alan when his clock shows k^2T. This is because according to the first postulate, Beryl's view of the universe is just as valid as Alan's. She sees time kT between passing Alan and receiving the signal (which he sent at his time T) so Alan must see time $k(kT)$ between passing Beryl and receiving her signal (which she sent at her time kT). It follows from this that Alan will assume that Beryl is at a distance $\frac{1}{2}(k^2-1)Tc$ away from him when she received the signal, as the light took $(k^2-1)T$ to get there and back and travelled at speed c. Furthermore Alan must assign the time $\frac{1}{2}(k^2+1)T$ to the moment when Beryl receives the signal, as he must assume that the light takes as long to get there as to come back. If Alan attempts to make an allowance for Beryl's velocity then he would have to assign different times to Beryl's response and to the response of a third observer moving with different velocity but answering from the same place at the same time — not a useful way of measuring the universe.

Thus, according to Alan, Beryl traveled a distance $\frac{1}{2}(k^2-1)Tc$ in time $\frac{1}{2}(k^2+1)T$, and so she has speed

$$v = \frac{k^2-1}{k^2+1}c. \tag{1}$$

This must also be the speed of Alan as measured by Beryl. Note from this that v is always between $-c$ and $+c$: an object cannot move faster than light. The light signals would not reach it if it did, which spoils our system of measurement; such an object (if it exists) would be invisible to us.

We solve equation (1) for k and find that

$$k = \sqrt{\frac{c+v}{c-v}}. \tag{2}$$

If we change v to $-v$ then k changes to $1/k$, a fact we will use later.

We further note from these calculations that the time that elapsed (according to Alan) between passing Beryl and her reception of the signal is $\frac{1}{2}(k^2+1)T$, but to Beryl this time gap is kT. So if Alan is watching Beryl doing experiments, then whenever Beryl measures a time gap of T, Alan must assign a time gap greater than T (after allowing for the travel time of light) by a factor of

$$\frac{k^2+1}{2k} = \frac{1}{2}\left(k+\frac{1}{k}\right) = (1-v^2/c^2)^{-1/2} \geq 1 \tag{3}$$

using equation (2). This latter expression is usually called γ, *the gamma factor*. Equation (3) shows how time is dilated by motion: moving clocks run slower by a factor of γ. That is, the clock that Beryl is using will seem to Alan to be running slowly. Of course the situation is symmetric: Beryl will see Alan's clock run slowly with respect to her clock at the same rate.[1]

This effect is not just an artefact of our choice of measuring systems, it is a real effect. Certain types of elementary particles called muons are created in the upper atmosphere of the earth and are detected on the ground. However, their lifetime when at rest is so short that even if they had travelled at the speed of light to pass through

[1] See, for example, J.C. Hafele and R.E. Keating, 'Around-the-World Atomic Clocks: Predicted Relativistic Gains', *Nature* Vol. 177 (1972) 166-170.

the atmosphere, they would have got less than one tenth of the way through before decaying — except that time dilation 'keeps them young'. Similar experiments done in particle accelerators with muons moving at speeds where γ is about 29 (99.88% of the speed of light) confirm the predictions of Special Relativity.

Simultaneity

We see from the above that requiring Alan and Beryl to measure time by their own clocks (a harmless enough requirement) and to use light signals to measure distances (radar would be equivalent, as radio waves are another form of the same sort of radiation) has lead us to conclude that the rate that time passes is not universal, but depends on how fast you are going. Even more remarkable is the fact that the very order of distant events can be different for different people. Consider figure 2, which shows Alan and Beryl again going past each other at a time both call $T = 0$.

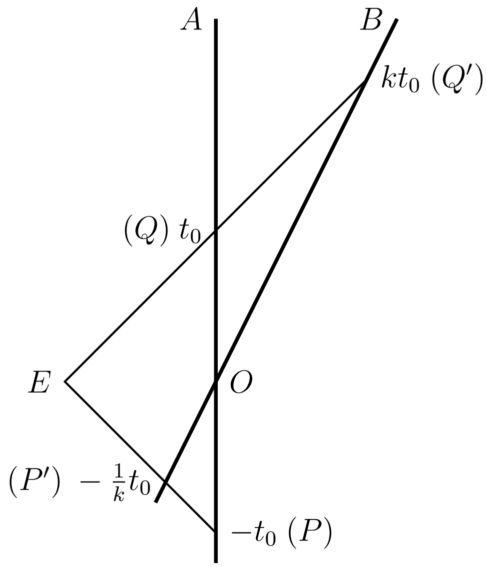

Figure 2: Simultaneity

The event E occurs at some distance from Alan. He sends out a light signal at time $-t_0$ (P on the diagram) that bounces off E and returns the signal, which reaches him at time t_0 (Q on the diagram). By the same reasoning used earlier, he must assign time $T = 0$ to the event E. So for Alan, E happened at the same time as he passed Beryl.

Now consider Beryl. If she is to do the same measurement of the time of E, she must send out a signal when Alan's signal reaches her (at P' on the diagram), and she will receive the bounced signal at Q'. We can use the k factor introduced earlier to find the time on Beryl's clock when she sends her signal and when she receives it. Assuming that Beryl moves with the same speed v before and after passing Alan we find that she measures a time gap of $(k - k^{-1})t_0$ between P' and Q', k as in equation (1).

Thus according to Beryl, event E happened at time

$$\frac{1}{2}\left(k - \frac{1}{k}\right) t_0 > 0.$$

Thus for Beryl, E happened *after* she passed Alan.

Obviously, if a third observer went past Alan (and Beryl) at $T = 0$ going in the opposite direction to Beryl, that observer would measure E as happening *before* the three met. What we get from this is that if events occur at different places then the order they occur in depends on the observer, or in the usual jargon *simultaneity is relative*.

The Twin Paradox

Suppose Alan and Beryl are twins, and Beryl goes off on a long journey through space at high speeds, and then returns to Alan who has been an inertial observer while she is away. According to time dilation, Beryl will have experienced much less time than Alan and will, as a consequence, be physically younger when she returns. How much younger will depend on her speed and could be calculated using γ.

Now comes the apparent paradox. According to Relativity's most basic rule Alan could just as well argue that Beryl stayed at rest, but he went on the trip: why is he older then Beryl when they meet again?

The problem here is that we are trying to apply the first postulate to a case where it cannot apply. In order to undertake her trip, Beryl must have accelerated away from Earth, and then accelerated again to come back (and a third time to land on Earth). All these accelerations mean she cannot have stayed inertial: there is no symmetry between the twins.

Neither can we argue that we could cut out the initial and final accelerations by sychronising clocks that go past each other: if one clock is to make each journey, the one that stays young must experience some sort of acceleration. It is also argued that we could keep the acceleration either very gentle or very brief, so that the youthful clock is either nearly inertial or inertial for all but a brief time. However the first idea is like claiming that a large semi-circle is nearly straight and so should have nearly the same length as a diameter. The second is, as Hermann Bondi pointed out, like the case of two drivers who go from A to B, the first in a straight line and the second in a wide deviation to P as in Figure 3. Both lines are straight almost everywhere, but one is clearly longer than the other.

Measuring Length

Let us now consider what Special Relativity tells us about lengths. Suppose we are an inertial observer, and are watching a rocket moving at speed v metres per second towards us from a distance we think is L metres away. Since distance is speed $\times$ time, the rocket will reach us after L/v seconds. However, time dilation means that the clocks on the rocket are running slowly, and so if they were reading 0 when the rocket was Lm away, they are reading $L/(v\gamma)$ when they reach us.

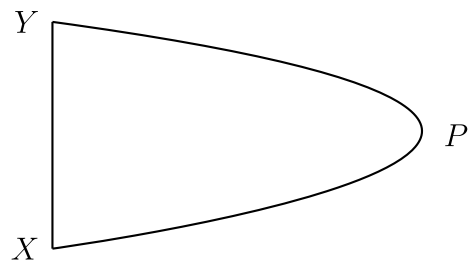

Figure 3: Two lines of nearly the same length?

Now the same thing must be observed by anyone on the rocket: from their point of view we have approached them at speed v, and have taken $L/(v\gamma)$ seconds to reach them. It follows that, if relativity is to be consistent, that from their point of view we did not start Lm away, but at a distance of L/γ. In other words, when moving distances are *shortened* by a factor of γ.

Just as in the case of time dilation, this effect is not just an accident of our way of measuring, it is a real effect. Unfortunately, no experimental verification of this **length contraction** has actually been done yet. However we see from the above argument that time dilation and length contraction must both occur if one does, or we are led to contradictions, and time dilation has been measured.

For historical reasons, length contraction is sometimes called **FitzGerald-Lorentz contraction**, and James Coleman in *Relativity for the Layman* quotes the limerick:

There was a young fellow called Fisk
Whose fencing was exceedingly brisk;
So fast was his action
The FitzGerald contraction
Turned his rapier into a disc.

Interstellar Travel

Now that we have the two ideas of time dilation and length contraction, we can explain how it is (in theory) possible to travel to a distant star, say 200 light years away, within a human lifetime. Recall that a light year is the distance that light travels in one year, about 6 million million miles (or $9\frac{1}{2}$ million million kilometres). As nothing can travel faster than light, how could we hope to get to such a star from earth within one lifetime?

Both time dilation and length contraction supply the answer. Suppose that a plucky astronaut makes the trip at a speed with $\gamma = 4$, that is at approximately 96.8% of the speed of light (about 289 thousand kilometres a second). From the point of view of the earth (which we take as inertial for the sake of argument), the astronaut takes 206.5 years to make the trip. However, time dilation means that the astronaut only experiences a quarter of this time span, or 51.64 years.

On the other hand, as far as the astronaut is concerned his clocks are running perfectly (the clocks on earth seem slow though), but the distance he has to travel is not 200 light years, but only 50. At 96.8% of the speed of light such a trip will take 51.64 years.

We see that if distances did not contract due to motion, we would be left with no way to resolve the conflict between the time the astronaut measures he takes for his trip and the fact that nothing can travel faster then light. If the distance to the star were not shrunk for the astronaut, then he would have travelled 200 light years in just over fifty of his years, that is at nearly four times the speed of light (as he measures it). This is not possible in relativity.

The Pole in the Barn Paradox

Now we know about length contraction, we can invent some amusing uses of it.

Suppose you want to fit a 20m pole into a 10m barn. If the pole were moving fast enough, then length contraction means it would be short enough. For the figures we have here, we need $\gamma = 2$, and that works out at a speed of about 86.6% of the speed of light, or 259.8 million metres per second.

Now comes the paradox. According to your friend who is going to slam the barn doors shut just as the end of the pole goes in, the pole is 10m long, and therefore it fits. However as far as you are concerned, the pole is still 20m long but the barn is now only 5m long: length contraction must work both ways by the first postulate. How can you fit this 20m pole into a 5m barn? This paradox is apparently due to Wolfgang Rindler of the University of Texas at Dallas.

Of course the key to this is relativity of simultaneity. Your friend sees the front end of the pole hit the back wall of the barn at the same time as the doors are closed, but you (and the pole) do not see things this way. You are standing still and see a 5m long barn coming towards you at some shockingly high speed. When the back of the barn hits the front of the pole (and takes the front of the pole with it), the back end of the pole must still be at rest. It cannot 'know' about the crash at the front, because the shock wave travelling along the pole 'telling it' about the crash travels at some finite speed. The front of the barn has only 15m to go to get to the back of the pole, but the shock wave has to travel the whole length of the pole, namely 20m. The speed of the barn is such that even if this shock wave travelled at the speed of light, it would not get to the back of the pole before the front of the barn did. Hence in both frames of reference, the pole fits inside the barn (and will presumably shatter when the doors are closed).

An important point to take from this is that if we get one result from correctly reasoning as one observer, then the same result must be true to any other inertial observer: we may need different reasoning though.

The Magician's Assistant Paradox

We can use the same principle to resolve another paradox.

The Magician's Assistant Paradox is the following (which I first had outlined to me by John Pulham of the University of Aberdeen): A magician is performing a trick with two guillotines set 160cm apart. Her assistant (who is 2m tall) is sent towards the guillotines, while lying down on a trolley. This trolley is moving at a fraction over 60% of the speed of light, so γ is slightly greater than 1.25. The magician drops the blades so that they fall when the assistant is exactly between the guillotines, which will miss him as length contraction makes him slightly less than 160cm tall to the magician and the guillotines. The blades then drop out of the assistant's way and he continues on unharmed.

But from the point of view of the assistant, the two blades are rushing towards him at 60% of the speed of light and rather than being 160cm apart are less than 128cm apart, and are therefore likely to cut him into three pieces.

We can resolve this paradox by explaining exactly what the assistant sees when the blades fall. Once again, it is the relativity of simulteneity that saves the assistants neck: one blade must fall before the other. In fact, we can see that the blade furthest from him must fall first (just in front of the top of his head), so that he goes *over* this blade in both his frame and the magician's. Then just as his feet pass the second blade, that one falls, and he goes *under* this blade in both his frame and the magician's. If you do the exact calculations, you find that this is indeed what happens.

Conclusions

What we have been discussing in this article is the effect that motion can have on the rate of a clock (or any other measurement of time) and on length and distance. The mathematics is simple, but the ideas behind what we are doing are subtle, and in places rather sophisticated. The key points are that, when moving, clocks are slowed and distances are shrunk. In both cases, this effect is only apparent in comparison to another inertial observer, and is symmetric.

If you are interested in relativity, there are many books on the subject: I would particularly recommend Hermann Bondi's *Relativity and Common Sense* (recently reprinted by Dover) and Wolfgang Rindler's *Introduction to Special Relativity*.

For further details, as well as discussion of other famous paradoxes and refutation of various false beliefs about relativity, start at John Baez' pages on the web, for example

www.math.ucr.edu/home/baez/relativity.html

There are many web sites dealing with Special and General Relativity, but you should be careful of the large number of alternate theories (one for each alternate theorist), as all of those I've seen are not worth persuing.