Parabola Volume 43, Issue 1 (2007)

Problems Section

You are invited to submit solutions to any or all of the following problems, accompanied by your name, (school and year if appropriate). Solutions to these problems will appear in the next issue of *Parabola*, and if received in time your solution(s) may be used.

Q1231 Given $a > 0$, prove that

$$\underbrace{\sqrt{a+\sqrt{a+\cdots+\sqrt{a}}}}_{n \text{ times}} < \frac{1+\sqrt{4a+1}}{2}.$$

Q1232 Let a and c be two distinct real numbers and b be their arithmetic average (i.e. $b = (a+c)/2$). Find the condition on a and c so that $\sin^2 a$ and $\sin^2 c$ are distinct, and that $\sin^2 b$ is the arithmetic average of $\sin^2 a$ and $\sin^2 c$.

Q1233 Prove that for any odd interger $n \geq 3$ and any $a \neq 0$ there holds

$$\left(1+a+\frac{a^2}{2!}+\frac{a^3}{3!}+\cdots+\frac{a^n}{n!}\right)\left(1-a+\frac{a^2}{2!}-\frac{a^3}{3!}+\cdots-\frac{a^n}{n!}\right) < 1.$$

Q1234 (suggested by J. Guest, Victoria; edited)

Is the following statement true? “No prime number of type $10s + 1$ is a divisor of any number of the type $5^n + 1$, where s and n are positive integers.” Give a reason for your answer.

Q1235 (suggested by J. Guest, Victoria) As Jack and Tony went for a stroll they spotted a nice fruit shop. Jack decided to buy 7 bananas, 3 oranges and 5 plums, while Tony bought 5 bananas, 7 oranges and 3 plums. Jack spent \$3.49 for his purchase and Tony \$3.89. You are given that the bananas cost more than 20 cents each. Find the price of all three types of fruit.

Q1236 Calculate the sum

$$6 + 66 + 666 + \cdots + \underbrace{66\cdots66}_{n \text{ 6s}}, \quad n \geq 1.$$

Q1237 Prove that the numbers 49, 4489, 444889, ..., obtained by inserting 48 in the middle of the preceeding number are all perfect squares.

Q1238 Let x, y and z be three positive numbers such that $x < y < z$. Prove that if $1/z$, $1/y$ and $1/x$ form an arithmetic progression then $z - x$, y, and $x - y + z$ are the lengths of the sides of a right-angled triangle.

Q1239 Find all real numbers x and y satisfying

$$4^{\sin x} - 2^{1+\sin x} \cos(xy) + 2^{|y|} = 0.$$

Q1240 Prove that if $S(x) = ax^2 + bx + c$ is an integer when $x = 0$, $x = 1$ and $x = 2$, then $S(x)$ is an integer whenever x is an integer.