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Solutions to Problems 1221–1230

Q1221 (submitted by Frank Drost, Research Associate, School of Mathematics and Statistics, UNSW. Edited.)

Complete the mathematical equations below by inserting the least number of mathematical symbols from the table

$+$	$-$	$\times$	$/$	$\sqrt{}$	!	(	)

on the left-hand side of the equation.

$$\begin{array}{ccccc} 0 & 0 & 0 & = & 6 \\ 1 & 1 & 1 & = & 6 \\ 2 & 2 & 2 & = & 6 \\ 3 & 3 & 3 & = & 6 \\ 4 & 4 & 4 & = & 6 \\ 5 & 5 & 5 & = & 6 \\ 6 & 6 & 6 & = & 6 \\ 7 & 7 & 7 & = & 6 \\ 8 & 8 & 8 & = & 6 \\ 9 & 9 & 9 & = & 6 \\ 10 & 10 & 10 & = & 6 \end{array}$$

ANS:

$$\begin{aligned} (0!+0!+0!)! &= 6 \\ (1+1+1)! &= 6 \\ 2+2+2 &= 6 \\ 3\times 3-3 &= 6 \\ 4+4-\sqrt{4} &= 6 \\ 5+5/5 &= 6 \\ 6+6-6 &= 6 \\ 7-7/7 &= 6 \\ \sqrt{8+8/8!} &= 6 \\ 9-9/\sqrt{9} &= 6 \\ \sqrt{10-10/10!} &= 6 \end{aligned}$$

Q1222 How many ways can n cards be dealt to two persons, given that they may receive unequal numbers of cards but each has at least one card?

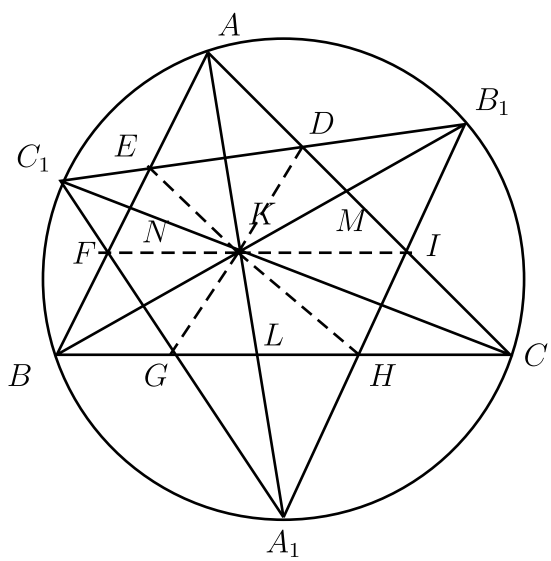
ANS: Let the two persons be A and B. For each card, there are 2 ways to deal: either A gets the card or B gets the card. So for n cards there are 2^n ways. But these include the two cases when A or B gets all the cards. So there are $2^n - 2 = 2(2^{n-1} - 1)$ ways to deal so that each person has at least one card.

Q1223 The three angle bisectors of a triangle ΔABC cut its circumcircle at A_1, B_1 and C_1. Let S be the common area of ΔABC and $\Delta A_1B_1C_1$. Prove that

$$S \geq \frac{2}{3}\text{area}(\Delta ABC).$$

When does equality occur?

ANS:



First we note that $\angle BB_1A_1 = \angle BAA_1 = \angle A_1AC$, so that

$$\angle AKB_1 = \angle KB_1A_1 + \angle KA_1B_1 = \angle A_1AC + \angle KA_1B_1 = \angle A_1IC = \angle AIB_1.$$

So $AKIB_1$ is a cyclic quadrilateral, which implies

$$\angle KAB_1 = \angle KIA_1. \tag{1}$$

On the other hand,

$$\begin{aligned}\angle KAB_1 &= \angle A_1AB_1 \qquad (2)\\ &= \angle A_1AC + \angle CAB_1\\ &= \angle A_1AB + \angle CBB_1\\ &= \angle A_1B_1B + \angle CBB_1\\ &= \angle CHB_1. \qquad (3)\end{aligned}$$

(1) and (3) give $\angle KIA_1 = \angle CHB_1$, implying $KI \parallel CH$. Similarly we can prove that $KH \parallel CI$, so that $CIKH$ is a parallelogram. In the same manner, we can prove that $AEKD$ and $BGKF$ are parallelograms. Therefore,

$$S(\Delta AED) = S(\Delta KDE),\ S(\Delta BGF) = S(\Delta KFG),\ S(\Delta CIH) = S(\Delta KHM). \qquad (4)$$

Now let $x = KL/AL$. Then it is easy to see that

$$x = \frac{S(\Delta KBC)}{S(\Delta ABC)}$$

(compare the heights of the two triangles). Also, since ΔABC and ΔKGH are similar triangles, we have

$$x^2 = \frac{S(\Delta KGH)}{S(\Delta ABC)}.$$

Similarly, if $y = KM/AM$ then

$$y = \frac{S(\Delta KCA)}{S(\Delta ABC)} \quad \text{and} \quad y^2 = \frac{S(\Delta KID)}{S(\Delta ABC)},$$

and if $z = KN/CN$ then

$$z = \frac{S(\Delta KAB)}{S(\Delta ABC)} \quad \text{and} \quad z^2 = \frac{S(\Delta KEF)}{S(\Delta ABC)}.$$

Therefore,

$$x + y + z = \frac{S(\Delta KBC) + S(\Delta KCA) + S(\Delta KAB)}{S(\Delta ABC)} = 1$$

and

$$\frac{S_1}{S(\Delta ABC)} = x^2 + y^2 + z^2,$$

where $S_1 = S(\Delta KGH) + S(\Delta KID) + S(\Delta KEF)$. By the Cauchy-Schwarz inequality there holds

$$1 = (x + y + z)^2 \le 3(x^2 + y^2 + z^2),$$

so that $\qquad S_1 \geq \frac{1}{3} S(\Delta ABC).$

Now if S is the common area between ΔABC and $\Delta A_1B_1C_1$ then, due to (4),

$$S_1 + 2(S - S_1) = S(\Delta ABC),$$

or

$$2S = S(\Delta ABC) + S_1 \geq \frac{4}{3} S(\Delta ABC),$$

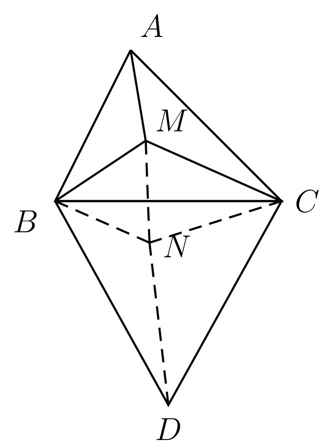
proving the desired inequality. Equality occurs when $x = y = z$, i.e. when ΔABC is equilateral.

Q1224 Let D be a point outside a triangle ΔABC such that A and D are on opposite side of the line BC, and that ΔBCD is equilateral. Prove that for any point M in the same plane with ΔABC

$$MA + MB + MC \geq AD.$$

When does equality occur?

ANS:



By rotating ΔABC 60° clockwise about B, the point C coincides with D and M with N. It is easy to see that

$$MA + MB + MC = AM + MN + ND \geq AD.$$

Equality occurs when M and N both lie on the line AD, i.e. M is the intersecting point of AD and the circumcircle of ΔBCD.

Q1225 (submitted by J. Guest, East Bentleigh, Victoria) Find all the real roots of

$$3x^5 - 40x^4 + 169x^3 - 271x^2 + 136x - 21 = 0.$$

ANS: (submitted by J. Guest)

The equation has an integral solution $x = 7$, so that

$$\begin{aligned}&3x^5 - 40x^4 + 169x^3 - 271x^2 + 136x - 21\\&\quad = (x-7)(3x^4 - 19x^3 + 36x^2 - 19x + 3) = 0.\end{aligned}$$

We now solve

$$3x^4 - 19x^3 + 36x^2 - 19x + 3 = 0,$$

which is a reciprocal equation. By dividing by x^2 and set $z = x + 1/x$ so that $x^2 + 1/x^2 = z^2 - 2$ we obtain

$$3z^2 - 19z + 30 = 0,$$

which has two solutions $z_1 = 3$ and $z_2 = 10/3$. The first value $z = 3$ leads to $x^2 - 3x + 1 = 0$ which has two real solutions $(3 \pm \sqrt{5})/2$. The second value $z = 10/3$ leads to $3x^2 - 10x + 3 = 0$ which has two real solutions $x = 3$ and $x = 1/3$. So all the real roots are

$$\frac{1}{3}, \quad \frac{3-\sqrt{5}}{2}, \quad \frac{3+\sqrt{5}}{2}, \quad 3, \quad 7$$

Q1226 Find the integral value of

$$\sqrt{6 + \sqrt{6 + \sqrt{6 + \sqrt{6 + \cdots}}}}$$

ANS: (submitted by J. Guest, Victoria)

Let

$$x = \sqrt{6 + \sqrt{6 + \sqrt{6 + \sqrt{6 + \cdots}}}},$$

so that

$$x^2 = 6 + x$$

or

$$(x-3)(x+2) = 0.$$

Since $x > 0$, the integral value to be found is 3.

Q1227 Consider n simultaneous equations in n unknowns $x_1, \ldots, x_n$:

$$\begin{aligned}x_1 + x_2 + x_3 &= 0\\x_2 + x_3 + x_4 &= 0\\\vdots \quad\quad &= 0\\x_{n-1} + x_n + x_1 &= 0\\x_n + x_1 + x_2 &= 0\end{aligned}$$

1. For which values of n do the equations have a unique solution?

2. Find the most general solution when the equations do not have a unique solution.

ANS: (submitted by John C. Barton, Victoria)

From the first two equations we deduce $x_1 = x_4$. From the second and third equations we deduce $x_2 = x_5$. Repeating the argument we obtain

$$x_1 = x_4 = x_7 = x_{10} = \cdots \tag{1}$$
$$x_2 = x_5 = x_8 = x_{11} = \cdots \tag{2}$$
$$x_3 = x_6 = x_9 = x_{12} = \cdots \tag{3}$$

and also

$$x_{n-2} = x_1, \quad x_{n-1} = x_2, \quad x_n = x_3. \tag{4}$$

If n is not a multiple of 3 then x_n is in group (1) or group (2), i.e.

$$(x_n = x_1 \quad \text{and} \quad x_{n-1} = x_3) \quad \text{or} \quad (x_n = x_2 \quad \text{and} \quad x_{n-1} = x_1).$$

Using (4) we deduce $x_1 = x_2 = x_3$, so that (noting that $x_1 + x_2 + x_3 = 0$)

$$x_1 = x_2 = \cdots = x_n = 0.$$

If n is a multiple of 3 then x_n is in group (3), and thus (1)–(4) reduce to (noting that $x_3 = -(x_1 + x_2)$)

$$\begin{aligned} x_1 &= x_4 = \cdots = x_{n-2} = a \\ x_2 &= x_5 = \cdots = x_{n-1} = b \\ x_3 &= x_6 = \cdots = x_n = -(a+b), \end{aligned} \tag{5}$$

for any real numbers a and b.

Therefore,

1. The system has a unique solution when n is not a multiple of 3, and the solution is $x_1 = x_2 = \cdots = x_n = 0$.
2. When n is a multiple of 3, the solutions are given by (5) for any real values of a and b.

Q1228 For any real number a, the symbol $[a]$ denotes the integer part of a. E.g.

$$[2] = 2, \quad [3.7] = 3, \quad [-2.4] = -3.$$

Simplify

$$[a] + \left[a + \frac{1}{n}\right] + \left[a + \frac{2}{n}\right] + \cdots + \left[a + \frac{n-1}{n}\right].$$

ANS: First we note that $a < [a] + 1 \le a + 1$, so that there exists an integer $k = 1, 2, \ldots, n$ satisfying

$$a + \frac{k-1}{n} < [a] + 1 \le a + \frac{k}{n}. \tag{1}$$

This implies

$$n[a] - k + n \le na < n[a] - k + n + 1,$$

which in turn gives

$$[na] = n[a] - k + n. \tag{2}$$

On the other hand, for any $j = 1, \ldots, k-1$,

$$[a] < a + \frac{j}{n} < [a] + 1,$$

so that

$$\left[a + \frac{j}{n}\right] = [a],$$

and for any $j = k, \ldots, n$,

$$[a] + 1 \le a + \frac{j}{n} \le a + 1,$$

so that

$$\left[a + \frac{j}{n}\right] = [a] + 1.$$

Therefore,

$$\begin{aligned}
[a] + \left[a + \frac{1}{n}\right] + \cdots + \left[a + \frac{n-1}{n}\right] \\
&= \underbrace{[a] + \cdots + [a]}_{k \text{ terms}} + \underbrace{([a]+1) + \cdots + ([a]+1)}_{n-k \text{ terms}} \\
&= k[a] + (n-k)([a]+1) \\
&= n[a] + n - k \\
&= [na]
\end{aligned}$$

where in the last step we use (2).

Q1229 A sequence of polynomials is defined recursively by

$$\begin{aligned}
F_1(x) &= \frac{x^2}{2} + \frac{x}{2} \\
F_k(x) &= k\left(\int_0^x F_{k-1}(t)\,dt + x\int_0^{-1} F_{k-1}(t)\,dt\right), \quad k \ge 2.
\end{aligned}$$

Find the constant term and the coefficients of x^k and x^{k+1} in $F_k(x)$.

ANS: (submitted by Julius Guest, Victoria)

Since $F_k(0) = 0$ for all $k \ge 1$, the constant term is 0. We will prove that the coefficient of x^k is $1/2$ and of x^{k+1} is $1/(k+1)$ by using induction on k.

The result is clearly true when $k = 1$. Assume that the result is true for $k = l - 1$, i.e. the coefficient in F_{l-1} of x^{l-1} is $1/2$ and of x^l is $1/l$. Then

$$\begin{aligned} F_l(x) &= l\left(\int_0^x F_{l-1}(t)\,dt + x\int_0^{-1} F_{l-1}(t)\,dt\right) \\ &= l\int_0^x \left(\frac{t^l}{l} + \frac{t^{l-1}}{2} + \text{ lower order terms }\right)dt + lx\int_0^{-1} F_{l-1}(t)dt \\ &= l\left(\left.\frac{t^{l+1}}{l(l+1)}\right|_0^x + \left.\frac{t^l}{2l}\right|_0^x + \text{ lower order terms }\right) \\ &= \frac{x^{l+1}}{l+1} + \frac{x^l}{2} + \text{ lower order terms }. \end{aligned}$$

By mathematical induction, the result is proved.

Q1230 Show that the polynomial $F_k(x)$ defined in the previous question satisfies

$$F_k(x) - F_k(x-1) = x^k.$$

ANS: We use induction again. It is clear that the result is true for $k = 1$. Assume that the result is true for $k = l - 1 \geq 1$, i.e.

$$F_{l-1}(x) - F_{l-1}(x-1) = x^{l-1}.$$

We prove that the result is true for $k = l$. We have from the definition of F_l

$$F_l'(x) = l\left(F_{l-1}(x) + \int_0^{-1} F_{l-1}(t)dt\right),$$

so that

$$F_l'(x-1) = l\left(F_{l-1}(x-1) + \int_0^{-1} F_{l-1}(t)dt\right),$$

implying

$$F_l'(x) - F_l'(x-1) = l\left(F_{l-1}(x) - F_{l-1}(x-1)\right) = lx^{l-1}$$

by the inductive assumption. Integrating both sides gives

$$F_l(x) - F_l(x-1) = x^l + c$$

for some constant c. Using $F_l(0) = 0$ (see **Q1229**) we find $c = 0$. By mathematical induction the result is proved for all $k \geq 1$.

Further notes on Q1212, Vol 42, No 3, 2006: Equality occurs in (3) when $x_0 = \pm 1$, and in (2) when

$$\frac{a}{x_0^3} = \frac{b}{x_0^2} = \frac{c}{x_0},$$

implying $x_0 = 1$ and $a = b = c$ or $x_0 = -1$ and $a = -b = c$. So there are four sets of values of x_0, a, b and c such that equality occurs in (1): $x_0 = 1$ and $(a, b, c) = \pm(2/3, 2/3, 2/3)$ or $x_0 = -1$ and $(a, b, c) = \pm(2/3, -2/3, 2/3)$. Among these only $(x_0, a, b, c) = (1, -2/3, -2/3, -2/3)$ and $(x_0, a, b, c) = (-1, 2/3, -2/3, 2/3)$ satisfy the given equation.