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Numerical Integration

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Numerical integration enables approximations to be found for $\int_a^b f(x)dx$ where the integral for $f(x)$ cannot be written in terms of elementary functions. Integration is the process of measuring the 'signed area' between the curve $y = f(x)$ and the x axis in between the end points $x = a$ and $x = b$. The 'signed area' is the area multiplied by $+1$ if $f(x)$ is above the x axis and multiplied by -1 if $f(x)$ is below the x axis. Integration is also anti-differentiation in the sense that if $f(x) = \frac{dF}{dx}$ then $\int_a^b f(x)dx = F(b) - F(a)$, a result known as the first fundamental theorem of calculus. This underpins the significance of integration in numerous applications. For example the area under the velocity-time graph gives distance travelled. A numerical method for evaluating $\int_a^b f(x)dx$ can thus be obtained using a quadrature formula

$$\int_a^b f(x)dx = \sum_{i=1}^{N} w_i.f(x_i)$$

based on approximating the area under the curve by the area of N rectangles of width w_i and height $f(x_i)$, as shown in the figure. The dot between the widths and heights in this formula and elsewhere in this article has been used for clarity to represent multiplication.

The formula is not exact for all functions $f(x)$ but the widths and heights can be chosen to make the formula exact for simple functions $f(x) = 1, x, x^2 \ldots$ as shown below.

$\mathbf{N = 1}$

How might we choose w_1 and $f(x_1)$ so that $\int_0^1 f(x)dx \approx w_1.f(x_1)$? First consider $f(x) = 1$, then $f(x_1) = 1$ and $\int_0^1 f(x)dx = \int_0^1 1\ dx = 1$.

By equating $\int_0^1 f(x)dx$ and $w_1.f(x_1)$, we find $w_1 = 1$. Now consider $f(x) = x$, then $f(x_1) = x_1$, and $\int_0^1 f(x)dx = \int_0^1 x\ dx = \frac{1}{2}$. By equating $\int_0^1 f(x)dx$ and $w_1.f(x_1)$, we now have $w_1.x_1 = \frac{1}{2}$. But since $w_1 = 1$, then $x_1 = \frac{1}{2}$. Therefore $\int_0^1 f(x)dx \approx f\left(\frac{1}{2}\right)$, which is the Midpoint Rule on $[0, 1]$. Now, consider the Midpoint Rule on $[a, b]$. Say,

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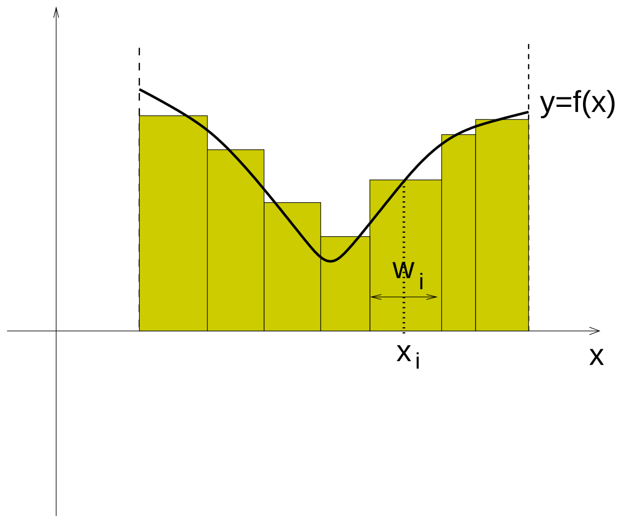

$\int_a^b f(x)dx \approx w_1 f(x_1)$. If $f(x) = 1$, then $\int_a^b 1\,dx = b - a$. Also, $w_1.f(x_1) = w_1.1 = w_1$ and hence $w_1 = b - a$. Consider $f(x) = x$, then $\int_a^b x\,dx = \frac{1}{2}(b^2 - a^2)$. Now, $w_1.f(x_1) = w_1.x_1$, but $w_1 = b - a$, so $(b - a).x_1 = \frac{1}{2}(b^2 - a^2)$. This last result can be solved to find x_1, viz:

$$\begin{aligned}(b-a).x_1 &= \frac{1}{2}(b-a)(b+a)\\ x_1 &= \frac{b+a}{2}\end{aligned}$$

Therefore,

$$\int_a^b f(x)dx \approx (b-a) f\left(\frac{a+b}{2}\right)$$

which is the Midpoint Rule on $[a, b]$. Note that the Midpoint Rule will integrate linear functions exactly.

N = 2

We now consider the quadrature rule

$$\int_0^1 f(x)dx \approx w_1 f(x_1) + w_2 f(x_2).$$

First take x_1 and x_2 as the end points then

$$\int_0^1 f(x)dx = w_1 f(0) + w_2 f(1).$$

Now, when $f(x) = 1$ then $\int_0^1 f(x)dx = 1$. That is,

$$\begin{aligned}w_1 f(0) + w_2 f(1) &= w_1.1 + w_2.1\\ &= w_1 + w_2\end{aligned}$$

(as $f(0)=f(1)=1$). So, $w_1+w_2=1$. Letting $f(x)=x$ we have $\displaystyle\int_0^1 f(x)dx=\frac{1}{2}$ and

$$\begin{aligned} w_1.f(0)+w_2.f(1) &= w_1.0+w_2.1 \\ &= w_2 \end{aligned}$$

(as $f(0)=0$ and $f(1)=1$). Therefore, $w_2=\dfrac{1}{2}$, which leads to $w_1=\dfrac{1}{2}$. Thus, $\displaystyle\int_0^1 f(x)dx=\frac{1}{2}[f(0)+f(1)]$. Now, consider the interval $[a,b]$ and

$$\int_a^b f(x)dx \approx w_1 f(a)+w_2 f(b).$$

When $f(x)=1$, $\int_a^b f(x)dx=b-a$. Also, $w_1 f(a)+w_2 f(b)=w_1+w_2$. Therefore, $w_1+w_2=b-a$. Now, when $f(x)=x$,

$$\int_a^b f(x)dx=\frac{1}{2}(b^2-a^2).$$

Also, $w_1 f(a)+w_2 f(b)=w_1 a+w_2 b$. Therefore, $w_1 a+w_2 b=\frac{1}{2}(b^2-a^2)$. Now we have the simultaneous equations

$$\begin{aligned} w_1+w_2 &= b-a, && (1) \\ w_1 a+w_2 b &= \frac{1}{2}(b-a)(b+a), && (2) \end{aligned}$$

to solve for w_1 and w_2.

Multiply equation (1) by a and subtract from equation 2, then

$$w_2=\frac{b-a}{2} \quad \text{and} \quad w_1=\frac{b-a}{2}.$$

This result yields the general form of the Elementary Trapezoidal Rule on $[a,b]$, i.e. $\displaystyle\int_a^b f(x)dx=\frac{b-a}{2}[f(a)+f(b)]$. This Rule will integrate linear functions exactly.

The Two-point Gauss Rule is an extension of the Trapezoidal Rule. This quadrature rule is also of the form:

$$\int_0^1 f(x)dx=w_1 f(x_1)+w_2 f(x_2),$$

but unlike the Trapezoidal Rule in which x_1 and x_2 are fixed at the ends of the interval, x_1 and x_2 are not predetermined. As there are now four unknowns, w_1, w_2, x_1 and x_2 we consider $f(x)=1, x, x^2, x^3$ to be exactly integrated. When $f(x)=1:$

$$\begin{aligned} \int_0^1 1\,dx &= 1 \\ &= w_1+w_2. \end{aligned}$$

When $f(x) = x$:

$$\begin{aligned}\int_0^1 x\ dx &= \frac{1}{2} \\ &= w_1.x_1 + w_2.x_2.\end{aligned}$$

When $f(x) = x^2$:

$$\begin{aligned}\int_0^1 x^2\ dx &= \frac{1}{3} \\ &= w_1.x_1^2 + w_2.x_2^2.\end{aligned}$$

When $f(x) = x^3$:

$$\begin{aligned}\int_0^1 x^3\ dx &= \frac{1}{4} \\ &= w_1.x_1^3 + w_2.x_2^3.\end{aligned}$$

In solving these four equations in four unknowns:

$$\begin{aligned}w_1 &= w_2 = \frac{1}{2} \\ x_1 &= \frac{1}{2} - \frac{1}{2\sqrt{3}} \\ x_2 &= \frac{1}{2} + \frac{1}{2\sqrt{3}}\end{aligned}$$

This gives the Two-point Gauss Rule:

$$\int_0^1 f(x)dx \approx \frac{1}{2}f\left(\frac{1}{2} - \frac{1}{2\sqrt{3}}\right) + \frac{1}{2}f\left(\frac{1}{2} + \frac{1}{2\sqrt{3}}\right).$$

When considering the general integral:

$$\int_a^b f(x)dx \approx w_1 f(x_1) + w_2 f(x_2).$$

The Two-point Gauss Rule can be derived similarly by considering the following integrals: For $f(x) = 1$:

$$\begin{aligned}\int_a^b 1\,dx &= b-a \\ &= w_1 + w_2 \\ \int_a^b x\,dx &= \frac{b^2-a^2}{2} \\ &= w_1x_1 + w_2x_2 \\ \int_a^b x^2\,dx &= \frac{b^3-a^3}{3} \\ &= w_1x_1^2 + w_2x_2^2 \\ \int_a^b x^3\,dx &= \frac{b^4-a^4}{4} \\ &= w_1x_1^3 + w_2x_2^3.\end{aligned}$$

The solutions to the simultaneous equations are:

$$\begin{aligned}w_1 &= w_2 = \frac{b-a}{2} \\ x_1 &= \left(\frac{b-a}{2}\right)\left(\frac{-1}{\sqrt{3}}\right) + \frac{b+a}{2} \\ x_2 &= \left(\frac{b-a}{2}\right)\left(\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}.\end{aligned}$$

Hence, $\displaystyle\int_a^b f(x)dx \approx$

$$\frac{b-a}{2} f\left(\frac{b-a}{2}\left(\frac{-1}{\sqrt{3}}\right) + \frac{b+a}{2}\right) + \frac{b-a}{2} f\left(\frac{b-a}{2}\left(\frac{-1}{\sqrt{3}}\right) + \frac{b+a}{2}\right).$$

N $= 3$

Simpson's Three Point (Elementary) Rule will now be considered. Say on the interval $[0,1]$ we choose $x_1 = 0,\ x_2 = \dfrac{1}{2}$ and $x_3 = 1$ for

$$\int_0^1 f(x)dx = w_1 f(x_1) + w_2 f(x_2) + w_3(x_3).$$

Then

$$\int_0^1 f(x)dx = w_1 f(0) + w_2 f\left(\frac{1}{2}\right) + w_3 f(1).$$

Values need to be chosen for w_1, w_2 and w_3 such that $1, x$ and x^2 can be exactly integrated. For $f(x) = 1$:

$$\begin{aligned}\int_0^1 f(x)dx &= 1 \\ &= w_1 + w_2 + w_3.\end{aligned}$$

For $f(x) = x$:

$$\begin{aligned}\int_0^1 f(x)dx &= \frac{1}{2} \\ &= \frac{1}{2}w_2 + w_3.\end{aligned}$$

For $f(x) = x^2$:

$$\begin{aligned}\int_0^1 f(x)dx &= \frac{1}{3} \\ &= \frac{1}{4}w_2 + w_3.\end{aligned}$$

Solving the three simultaneous equations:

$$w_1 = w_3 = \frac{1}{6} \quad \text{and} \quad w_2 = \frac{2}{3}.$$

Hence, the Simpson's Elementary Rule is given by:

$$\int_0^1 f(x)dx \approx \frac{1}{6}\left[f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right].$$

Simpson's Elementary Rule integrates quadratics as well as cubics exactly. If the interval $[a, b]$ is taken it can be shown that the Simpson's Elementary Rule is given by:

$$\int_a^b f(x)dx = \frac{b-a}{6}(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)).$$

To improve the accuracy of applications of the discussed rules, the number of sample points can be increased by deriving more complicated rules or by dividing the range into many subintervals.

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