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Nuclear power and the hairy doughnut

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Introduction

Many of you reading this article will be aware of the problems the world faces over energy supply: can we rely on fossil fuels (oil and coal), or should we look again at nuclear power? By nuclear power we usually mean **fission**, the break up of heavy atoms (uranium) to lighter ones. There is another type of nuclear power, though, **fusion**, where energy is released by combining light atoms (hydrogen) into heavier ones. This is the process that fuels the stars.

Fusion power offers the prospect of nearly unlimited fuel supply — hydrogen is the most abundant element in the Universe. Unfortunately, we don't seem to be very close to being able to make fusion produce power rather than consume it. I won't confess my age, but I have been hearing the claim that we will have working fusion power plants in 10-15 years for over 25 years.

The problem we face is that fusion only works if the fuel is at a very high temperature. The hydrogen then forms a **plasma**, where an electron is stripped off the nucleus. Stars form and control plasma using gravity, as their enormous mass compresses and heats the hydrogen enough to start fusion and keep the plasma contained. On earth, we cannot collect enough hydrogen to use gravity to make and contain plasma. So we form plasma by heating hydrogen to millions of degrees and confine it using magnetic fields, and this is where the hairy doughnuts of my title come in.

The devices used to confine plasma are called **tokamaks**, a Russian acronym, derived from the Russian words for 'toroidal chamber and magnetic coil'. They and are the shape of a doughnut, or what mathematicians call a **torus**. Why use such a shape? After all, stars are spherical.

To answer this question we need to look at the mathematical idealisation of magnetic fields, **vector fields**.

Vector Fields

The basic fact we know about magnetic fields is that they have a strength and a direction. If you've ever put a bar magnet under a piece of card and sprinkled iron filings onto the card, you've done the classic experiment to show the direction of the magnetic field around the magnet.

You can think of a vector field as a little arrow or hair at each point on some surface (or even over a region of space), the direction of the hair giving the direction of the vector field and the length its size, see figures 1 and 2. We also want vector fields to

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be continuous, which we can just take to mean that the length and direction do not undergo sudden jumps.

If we want to create a magnetic field to confine a plasma in a fusion reactor, then what we want is a vector field on a closed surface that we can keep the plasma inside. The vector field will then be pointing along the surface, or rather **tangent** to it: it is a **tangent vector field**. A tangent vector field lies on the surface, as if it is hair that has been combed flat. A familar example of a tangent vector field is the wind: wind has a direction and a strength, and it forms a vector field tangent to the surface of the earth.

Furthermore, for a fusion plant we want to make sure there is no place where the field vanishes, as that would allow the plasma to leak
out. A place where a vector field vanishes is called a **singularity** of the field.

So the problem we are faced with is the following: what closed surfaces can have a nowhere zero tangent vector field? Or, less formally, what surfaces can we comb?

There is a remarkable theorem that gives us the answer to this, the **Poincaré Index Theorem**. To state the theorem we need firstly to define something that allows us to count singularities, and this is the **index** of the theorem. The second thing we will need is something that seems to have nothing to do with vector fields: the **Euler characteristic**.

Index

The index of a vector field's singularity is an integer attached to the singularity, and it can be positive or negative. To measure it, draw a little circle around the singularity, small enough to ensure that only one singularity is inside the circle. (If this is not possible, then we cannot define the index, but these sorts of vector fields don't concern us.) Then as you go anti-clockwise around the circle, the vector field will change direction. The index is the number of complete revolutions the vector field makes counting anti-clockwise as positive and clockwise as negative.

In figure 1, we have a vector field whose singularity has index -1: as you go around the circle in the picture, the vector field turns exactly once clockwise.

See if you can work out the indices of the singularities in figure 2 (see end of article for the answer).

Euler characteristic

The **Euler characteristic** of a surface is a certain integer you attach to a surface. To calculate it, take your surface and split it up any way you like into polygons. You are allowed to have polygons with curved edges, but given any two polygons, either they never intersect, or if they do intersect, they do so only along *complete* common edges or *one* common vertex, see figure 3.

We assume you can cover the whole surface with a finite number of finite polygons. Suppose you have a total of F polygons (F for 'faces'), E edges and V vertices. The **Euler characteristic**, χ (the Greek letter 'chi') of your surface is then $\chi = V - E + F$. There are two important things to know about χ.

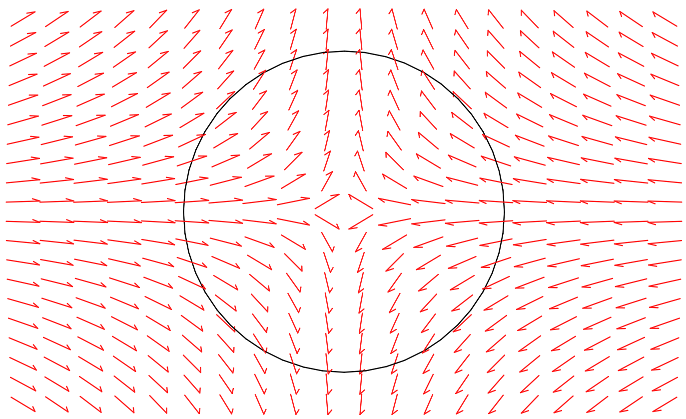

Figure 1: Index -1

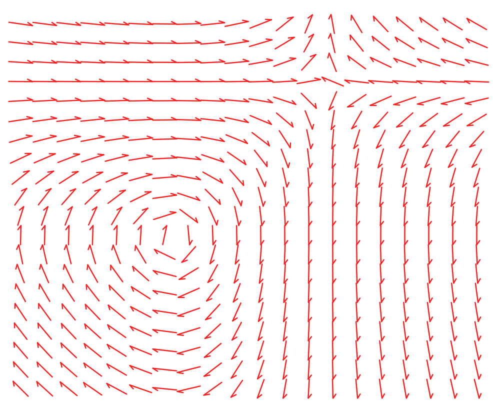

Figure 2: Two singularities

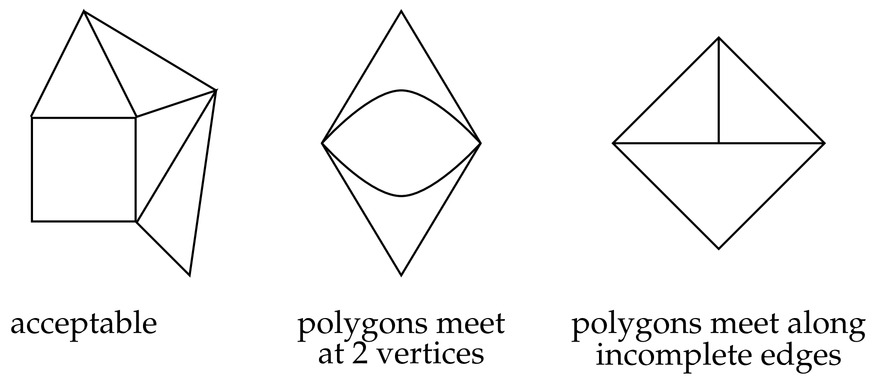

Figure 3: Polygonal Subdivisions

Firstly, it does not matter how you do the subdividing, you always get the same value of χ. To prove this, consider how we could change a subdivision. There are three ways we can do this 'refining'. Splitting an edge with a new vertex sends $V \to V+1$ and $E \to E+1$ leaving F unchanged. Secondly, adding a new edge between two vertices sends $F \to F+1$ and $E \to E+1$ leaving V unchanged. Finally adding a vertex and two edges changes $F \to F+1$, $V \to V+1$, $E \to E+2$. All these leave χ unchanged.

We see these in figure 4, where we alter the subdivision on the left in figure 3.

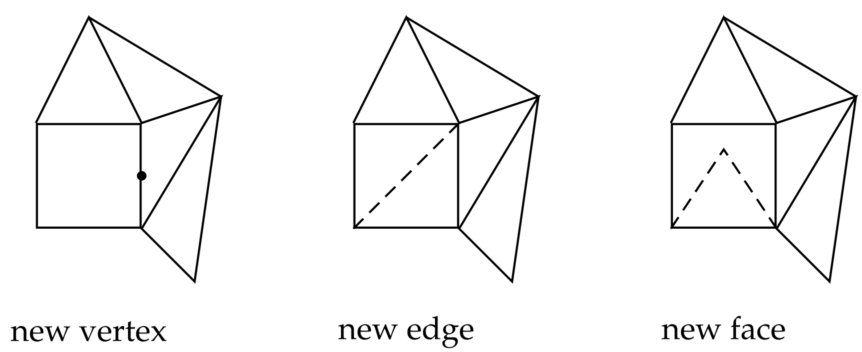

Figure 4: Refining Subdivisions

(Note that adding a new face breaks the 'one complete edge rule', but if you add some more edges you can make sure you satisfy the rule.)

If we now had two subdivisions, then we can create a common refinement of both of them by adding in vertices where an edge from the first intersects an edge from the second or vice versa. Doing so does not change χ at any stage. So the three divisions all have the same value of χ, and this proves the result.

Secondly, χ is an example of a **topological invariant**. This means that it stays the same even if we deform the surface (as long as we don't tear it), since the polygons will stay polygons, and we won't change the way the polygons intersect. We can even make sure the faces are all flat. As an example, consider a cube. If we have a cube made of rubber we can imagine blowing it up until it is the shape of a sphere. The square faces of the cube then form a subdivision of the sphere. But we know a cube has 6 faces, 12 edges and 8 vertices. So a cube has $\chi = 2$, and this must be the Euler characteristic of the sphere too.

Similarly, a soccer ball (or the buckminsterfullerine molecule C_{60}, correctly called a **truncated icosahedron**) has 32 faces (12 pentagons and 20 hexagons), 60 vertices and 90 edges: once again $\chi = 2$.

But what about the torus? Well, in figure 5 we have a torus-shaped box.

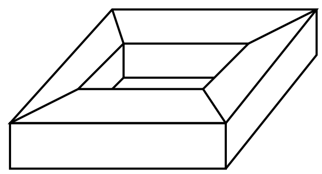

Figure 5: Toroidal Box

If you count you will see that there are 8 vertices each on the top and the bottom; four sets of 4 faces (top, bottom, inner faces, outer faces); two sets of twelve edges (top and bottom) and two sets of 4 edges (inner and outer faces). That is,

$$\chi = 2 \times 8 - (2 \times 12 + 2 \times 4) + 4 \times 4 = 0.$$

Among other things, this tells us that the sphere and the torus are topologically distinct: we cannot deform a sphere into a torus without tearing it.

See if you can prove that for a 'double-holed torus' we get $\chi = -2$. Be careful you form a valid subdivision.

Poincaré Index theorem

Now we can state the Poincaré Index theorem: *On a closed surface a tangent vector field can only have a finite number of singularities, and the sum of their indices must be the Euler characteristic of the surface.*

The remarkable thing about this result is that it links something that is calculated by counting polygons to something calculated by watching directions rotate.

We cannot really prove this result here, but we can see two immediate corollaries. The first is sometimes called the **Hairy Ball Theorem**: *It is impossible to comb the hair on a sphere*. This is because χ of a sphere is 2, so there must be *some* singularity in any tangent vector field on the sphere. The Hairy Ball theorem tells us that there must be at least some point on the earth's surface where the air is still. It is also why we cannot use a sphere to confine the plasma in a fusion reactor. Stars of course do not use magnetic confinement, so they are quite happy being spherical.

The second corollary is that the sum of the indices on a torus must be zero. This does not prove that there can be a singularity-free vector field on a torus, since the

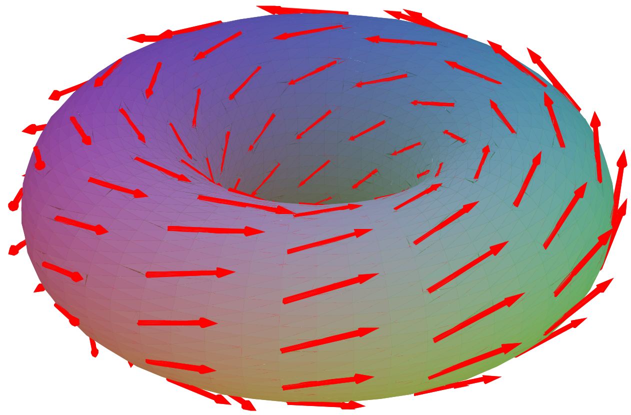

Figure 6: Spiral field on torus

index can be positive or negative. Fortunately, it is possible to have singularity-free vector fields on the torus: imagine a hairy doughnut lying flat on a plate, and comb it so that the hairs lie horizontally. In tokamaks they actually use a field that winds around the surface in a spiral, using the currents in the plasma itself to help the field to spiral around, see figure 6.

In closing, let me say that it was proved just over 100 years ago, that there are only two closed surfaces with $\chi = 0$. The torus is the *only* closed two-sided surface with $\chi = 0$, and so is the *only* shape we could possibly use to form a plasma containment vessel.

Answer to question

In figure 2 the lower singularity has index 1 and the upper one index -1.