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History of Mathematics: More on Ptolemy's Theorem

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I could kick myself! I have to begin this column by confessing to a stupid mistake. Here is what happened. I was surfing the net when I came upon a website that held great interest for me. What I should have done, of course, was to bookmark the page or at least take a note of the URL. But thinking that I could readily retrace my steps at some future date, I did neither, with the result that when I came to look for it again, I found myself lost completely in cyberspace. So now I can only give you the gist of the page that caught my attention.

The website seemed to be a blog from a US professor of Engineering. As I have lost his name, I shall simply call him Professor X. He raised several interesting questions in the context of a discussion of a class he had conducted. He had introduced the topic of Ptolemy's theorem, which was the subject of my last column, and found that none of his students had ever encountered it. Given that Euclidean geometry has been almost completely banished from school syllabuses, this is perhaps not entirely surprising. But he did find that almost all of the students in his class were able to test the result experimentally by, in one way or another, programming a computer to check its validity.

In order to make this article self-contained, I remind readers of what the theorem says. Look at Figure 1.

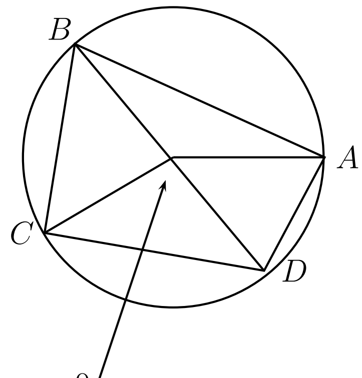

Figure 1

This shows a convex quadrilateral $ABCD$ whose vertices all lie on a circle with centre O. For ease of description, we set up the following notation:

$$|AB| = a$$
$$|BC| = b \qquad |AC| = x$$
$$|CD| = c \qquad |BD| = y$$
$$|DA| = d.$$

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Ptolemy's Theorem now says that $ac + bd = xy$.

I gave a proof of this result in my previous column, and will later give others in this one, but before I do so, let us return to Professor X's blog. It seems that each student generated a random concyclic quadrilateral (i.e. one fitting exactly into a circle as in Figure 1) and then had the computer calculate $ac + bd - xy$ for it. In every case, the result (to 12 significant figures) was zero.

Professor X was concerned to raise two matters: the first was to comment on the changing nature of Mathematics – the replacement of booklearning with an experimental approach; the second was a query – how much faith can we put in such an experimental result? Answering his own question, Professor X seemed to conclude that the experiment was as conclusive as the formal demonstration to be found in geometry texts. He asked how many such results would be needed to establish the result conclusively, and tentatively suggested the answer three. This is what led to my own interest in the problem.

My first approach was to use polar coordinates. Without any loss of generality, we may take the radius of the circle to be 1. Now establish some further notation that will also be useful later. Set:

$$\langle AOB = \alpha \qquad \langle BOC = \beta \qquad \langle COD = \gamma \qquad \langle DOA = \delta,$$

where $\alpha + \beta + \gamma + \delta = 2\pi$. We now need three angles to be chosen at random: α is to be taken as a random number in the interval $0 < \alpha < 2\pi$; $\alpha + \beta$ is to be taken as a random number in the interval $\alpha < \alpha + \beta < 2\pi$; $\alpha + \beta + \gamma$ is to be taken as a random number in the interval $\alpha + \beta < \alpha + \beta + \gamma < 2\pi$. [Note the use of radian measure; it is a good habit to adopt.] Thus the polar representations of the vertices may be taken (without any loss of generality) as:

$$A = (1, 0) \quad B = (1, \alpha) \quad C = (1, \alpha + \beta) \quad D = (1, \alpha + \beta + \gamma),$$

or, in the more familiar Cartesian co-ordinates,

$$\begin{aligned} A &= (1, 0) \quad B = (\cos\alpha, \sin\alpha) \quad C = (\cos(\alpha + \beta),\ \sin(\alpha + \beta)) \\ D &= (\cos(\alpha + \beta + \gamma),\ \sin(\alpha + \beta + \gamma)). \end{aligned}$$

Listed below are the results of three runs in Excel. Each block of figures represents a test run. All angles I have labelled *phi*. Thus in the first block the angular coordinates are respectively $0, \alpha, \alpha + \beta$ and $\alpha + \beta + \gamma$ and similarly for each of the other blocks. The second and third columns in each block provide the Cartesian coordinates of the points A, B, C, D respectively. From these the lengths of a, b, c, d, x and y are calculated. Finally, for each set, the value of $ac + bd - xy$ is derived and the result is (as it should be) zero in every case. The result is valid to six decimal places in all cases, although this accuracy could have readily been extended.

phi	sin(phi)	cos(phi)		lengths		products	
0	0	1	A	1.65223	AB,a	2.24844214	ac
1.944356	0.931034	-0.3649	B	1.005672	BC,b	1.56825355	bd
2.998109	0.142992	-0.9897	C	1.360853	CD,c	3.81669569	xy
4.494798	-0.97642	-0.2159	D	1.559409	DA,d		
				1.994855	AC,x		
				1.913269	BD,y	0	ac+bd-xy
0	0	1	A	1.843702	AB,a	2.20132699	ac
2.345659	0.714517	-0.6996	B	0.462958	BC,b	0.82325328	bd
2.812854	0.322849	-0.9465	C	1.193971	CD,c	3.02458027	xy
4.092331	-0.81384	-0.5811	D	1.778248	DA,d		
				1.973043	AC,x		
				1.532952	BD,y	0	ac+bd-xy
0	0	1	A	1.230839	AB,a	0.1309039	ac
1.325874	0.970156	0.24248	B	1.446191	BC,b	2.88927164	bd
2.942421	0.197857	-0.9802	C	0.106351	CD,c	3.02017554	xy
3.048822	0.092638	-0.9957	D	1.997849	DA,d		
				1.990091	AC,x		
				1.517607	BD,y	0	ac+bd-xy

This provides powerful evidence that the theorem holds good, and once it would have provided the spur towards the production of a purely formal proof. However, I take the thrust of Professor's question to be, 'With evidence like this, do we really need a formal proof at all?'

This is the matter I wish to discuss further.

My first approach to the question was to consider the possible ways in which such an experiment might be set up. Professor X gave little detail on this matter, but I soon realized that my approach was by no means the only one available.

Here is another. From the triangle OAB, we have $a = 2\sin\dfrac{\alpha}{2}$ and similarly for the other sides of the quadrilateral. We may likewise express x and y in terms of the angles. The full table is (with $\delta = 2\pi - \alpha - \beta - \gamma$):

$$a = 2\sin\frac{\alpha}{2} \quad b = 2\sin\frac{\beta}{2} \quad c = 2\sin\frac{\gamma}{2} \quad d = 2\sin\frac{\delta}{2}$$

$$x = 2\sin\frac{\alpha+\beta}{2} \quad y = 2\sin\frac{\beta+\gamma}{2}.$$

To prove Ptolemy's theorem, we therefore need to show (after clearing a common factor of 4) that

$$\sin\frac{\alpha}{2}\sin\frac{\gamma}{2} + \sin\frac{\beta}{2}\sin\frac{\delta}{2} = \sin\frac{\alpha+\beta}{2}\sin\frac{\beta+\gamma}{2} \tag{1}$$

where

$$\alpha + \beta + \gamma + \delta = 2\pi. \tag{2}$$

This is the sort of thing that would have been set as a trigonometry exercise in my high school days (although it would have been marked with an asterisk, to denote a problem harder than the usual). Here is how a proof could go:

$$\begin{aligned}
&\sin\frac{\alpha}{2}\sin\frac{\gamma}{2}+\sin\frac{\beta}{2}\sin\frac{\delta}{2}\\
= \;& \frac{1}{2}\left(\cos\frac{\alpha-\gamma}{2}-\cos\frac{\alpha+\gamma}{2}\right)+\frac{1}{2}\left(\cos\frac{\beta-\delta}{2}-\cos\frac{\beta+\delta}{2}\right)\\
= \;& \frac{1}{2}\left(\cos\frac{\alpha-\gamma}{2}+\cos\frac{\beta-\delta}{2}\right)-\frac{1}{2}\left(\cos\frac{\alpha+\gamma}{2}+\cos\frac{\beta+\delta}{2}\right)\\
= \;& \cos\frac{\alpha+\beta-\gamma-\delta}{4}\cos\frac{\alpha-\beta-\gamma+\delta}{4}\\
& \qquad +\cos\frac{\alpha+\beta+\gamma+\delta}{4}\cos\frac{\alpha-\beta+\gamma-\delta}{4}
\end{aligned}$$

$$\begin{aligned}
= \;& \cos\frac{\alpha+\beta-\gamma-\delta}{4}\cos\frac{\alpha-\beta-\gamma-\delta}{4} && \text{by Equation (2)]}\\
= \;& \cos\frac{2\pi-2\gamma-2\delta}{4}\cos\frac{2\pi-2\beta-2\gamma}{4}\\
= \;& \sin\frac{\gamma+\delta}{2}\sin\frac{\beta+\gamma}{2}\\
= \;& \sin\frac{\alpha+\beta}{2}\sin\frac{\beta+\gamma}{2} && \text{[again by Equation (2)].}
\end{aligned}$$

This provides a proof of Ptolemy's theorem (and incidentally reinforces my remarks in my previous column about the construction of chord tables as a precursor to Trigonometry), but it doesn't address Professor X's question. However, the analysis just given may be modified to do just that. Consider

$$\sin\frac{\alpha}{2}\sin\frac{\gamma}{2}+\sin\frac{\beta}{2}\sin\frac{\delta}{2}-\sin\frac{\alpha+\beta}{2}\sin\frac{\beta+\gamma}{2}=E\text{ (say)}$$

and expand this expression in terms of α. Keep β and γ fixed, but remember that δ is given by $\delta=2\pi-\alpha-\beta-\gamma$. We then have

$$E=\sin\frac{\alpha}{2}\sin\frac{\gamma}{2}+\sin\frac{\beta}{2}\sin\frac{2\pi-\alpha-\beta-\gamma}{2}-\sin\frac{\alpha+\beta}{2}\sin\frac{\beta+\gamma}{2}.$$

Now continue with the expansion of E in terms of $a = 2\sin\frac{\alpha}{2}$. We find

$$
\begin{aligned}
E &= \sin\frac{\alpha}{2}\sin\frac{\gamma}{2} + \sin\frac{\beta}{2}\sin\left(\pi - \frac{\alpha+\beta+\gamma}{2}\right) \\
&\qquad - \sin\frac{\beta+\gamma}{2}\left(\sin\frac{\alpha}{2}\cos\frac{\beta}{2} + \cos\frac{\alpha}{2}\sin\frac{\beta}{2}\right) \\
&= \sin\frac{\alpha}{2}\left(\sin\frac{\gamma}{2} - \sin\frac{\beta\gamma}{2}\cos\frac{\beta}{2}\right) + \sin\frac{\beta}{2}\sin\left(\frac{\alpha+\beta+\gamma}{2}\right) \\
&\qquad - \cos\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\beta+\gamma}{2} \\
&= \sin\frac{\alpha}{2}\left(\sin\frac{\gamma}{2} - \sin\frac{\beta+\gamma}{2}\cos\frac{\beta}{2} + \sin\frac{\beta}{2}\cos\frac{\beta+\gamma}{2}\right) \\
&\qquad + \cos\frac{\alpha}{2}\left(\sin\frac{\beta}{2}\sin\frac{\beta+\gamma}{2} - \sin\frac{\beta}{2}\sin\frac{\beta+\gamma}{2}\right) \\
&= \sin\frac{\alpha}{2}\left(\sin\frac{\gamma}{2} - \sin\frac{\beta+\gamma}{2}\cos\frac{\beta}{2} + \sin\frac{\beta}{2}\cos\frac{\beta+\gamma}{2}\right).
\end{aligned}
$$

At this point, we can work on the coefficient of $\sin\dfrac{\alpha}{2}$.

$$\sin\frac{\beta}{2}\cos\frac{\beta+\gamma}{2} - \sin\frac{\beta+\gamma}{2}\cos\frac{\beta}{2} + \sin\frac{\gamma}{2} = \sin\frac{-\gamma}{2} + \sin\frac{\gamma}{2} = 0.$$

Thus we have once again proved that $E = 0$, and in truth this proof is merely a recasting of the previous one.

However, there is another way to proceed. Just beyond the half-way point in the above derivation, we reached a point where we had

$$E = \sin\frac{\alpha}{2}F_1(\beta,\gamma) + \cos\frac{\alpha}{2}F_2(\beta,\gamma),$$

where $F_1(\beta,\gamma)$ and $F_2(\beta,\gamma)$ were complicated expressions involving β and γ. (Both ultimately turned out to be zero, but suppose we got lazy and wanted to avoid the work involved in proving this.) Recall that $\sin\frac{\alpha}{2} = \frac{\alpha}{2}$ so that $\cos\frac{\alpha}{2} = \sqrt{1-\frac{a^2}{4}}$. The aim is to show that $E = 0$. That is to say, we want to establish the result

$$aF_1(\beta,\gamma) = -\sqrt{1-\frac{a^2}{4}}F_2(\beta,\gamma),$$

which we may write as

$$4\left((F_1(\beta,\gamma)^2 + (F_2(\beta,\gamma)^2\right)a^2 - 4\left(F_2(\beta,\gamma)\right)^2 = 0.$$

The left-hand side of this equation is a quadratic in a, and so is the right, albeit a very simple one!

Now, let me introduce a very powerful theorem. It is not as well-known as it deserves, but it can be most useful. It states that:

If $P_N(x)$ and $Q_n(x)$ are two polynomials of degree n in x, and if $P_n(x) = Q_n(x)$ for $n+1$ different values of x, then $P_n(x)$ and $Q_n(x)$ are identically equal.

This provides the method of proof known as 'pseudo-induction'. We seek to apply it here, and because the two polynomials in our case are quadratics, we will have proved the result for all cases if we can find three cases for which it holds. Because every value of a corresponds to a value of α and *vice versa*, it will suffice, in order to prove Equation (1) to show that it holds for *three* different values of α. So it would seem that Professor X had a certain serendipity on his side!

Furthermore, if we can verify Equation (1) for three different values of α, then this constitutes a complete and rigorous proof. What Professor X was proposing is what is called a *heuristic*. The Mathworld website gives as one definition of this word 'convincing without being rigorous'. Certainly this meaning is most apt in this context. However, to complete the proof, we do need to demonstrate three values of α that lead to cases in which Equation (1) holds.

An obvious candidate is $\alpha = 0$, i.e. $a = 0$. I leave the details to the reader, but remind that reader that in this instance $\beta + \gamma + \delta = 2\pi$. Exactly the same argument applies in the case $d = 0$, which corresponds to the case $\alpha = 2\pi - \beta - \gamma$. What is probably the simplest choice for the third value is $\alpha = \beta + \gamma$. This results in a somewhat simpler version of an earlier analysis (the reader may fill in the details) . So here we have another proof of Ptolemy's theorem.

I close with a further observation and a reminiscence.

The observation provides a further heuristic, but one which means that we need only check *one* instance in order to provide a convincing (albeit incompletely rigorous) demonstration of the theorem. In order to do this, we relax an assumption I made earlier. I took the radius of the enclosing circle to be 1; now let it be r. The result of this change is merely to alter the scale of the underlying diagram (Figure 1). All the lengths will now be exactly r times as long as they were in the earlier case.

Now consider xy. This will be given by some formula F involving a, b, c, d.

$$xy = F(a, b, c, d).$$

Scaling the diagram by a factor of r results in a scaling of the product xy by a factor of r^2, which suggests that the required formula involves products of lengths: some or all of the products $a^2, b^2, c^2, d^2, ab, ac, ad, bc, bd, cd$. Moreover the formula will need to exhibit the geometric symmetries inherent in the problem:

- Interchange of a and c leaves the formula unchanged, as this merely rearranges the way in which the diagram is labeled
- Interchange of b and d leaves the formula unchanged, for the same reason
- Interchange of the pair (a, c) with the pair (b, d) leaves the formula unchanged, again for a similar reason.

The first two considerations suggest that the formula we need involves the products in the combinations $ac, a^2 + c^2, bd, b^2 + d^2$, all of which display the required symmetries. The third requirement suggests that it involves them either as $ac + bd$, or else as $a^2 + b^2 + c^2 + d^2$. However, this latter possibility would leave the formula unchanged if a and b were interchanged, which is clearly wrong because this interchange *does* materially alter the diagram. So we are led to the formula

$$xy = F(ac + bd),$$

indeed to the formula

$$xy = K(ac + bd),$$

where K is a constant. The reason for this is that such a right-hand side scales as r^2 exactly as required.

We can now determine the value of K by simply considering *one* special case. A good one to consider is that of a square $ABCD$, for which

$$a = b = c = d; \qquad x = y = d\sqrt{2}.$$

From this case, we see that $K = 1$, and Ptolemy's theorem results.

This argument is not a rigorous proof, although it may well be possible to construct one along these lines by extending the ideas it entails. It is however a powerful *heuristic*. (I will say more about such arguments in my next column.)

The reminiscence I use to close this article does not involve Ptolemy's theorem, but speaks nevertheless to the power of heuristic arguments. About 20 years ago, I had a letter and a bulky manuscript arrive on my desk. The author was a man, Mr Y, I will call him, who lived in a third world country, and would seem to have been an amateur mathematician.

He had developed a new method of finding approximate zeroes of polynomials. There are several such methods already available, so that there is no great need for another, especially as his approach only applied to a very limited class of problems. Nor were these the only difficulties. His 'proof' that the method worked consisted of the production of numbers of enormous algebraic formulae, all true, but he then presumed that a pattern that he discerned in their appearance would persist whenever they were generalized. This was by no means obvious.

I wrote to Mr Y pointing out, as nicely as I could, that these difficulties existed. However, I was greatly impressed that his formulae produced values that were accurate to 10 decimal places! I checked the examples he provided using the computer algebra package MAPLE, and also by means of a powerful hand–held calculator. Both these confirmed the results Mr Y claimed. I thought: 'There must be something in this, even though the supporting argumentation is inadequate'. For a while, I tried to see what this something was, but got nowhere, and so directed my attentions elsewhere.

Then just last year, when I was looking for something completely different, my eye lighted on a passage in an old textbook of Calculus. A theorem was proved there which provided *exactly* what Mr Y needed to prove his case. I wrote to Mr Y, but my letter seemed to fall into a black hole. It would have been nice to have him see the validation of his claim, but alas this seems not to have happened.