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UNSW School Mathematics Competition 2007

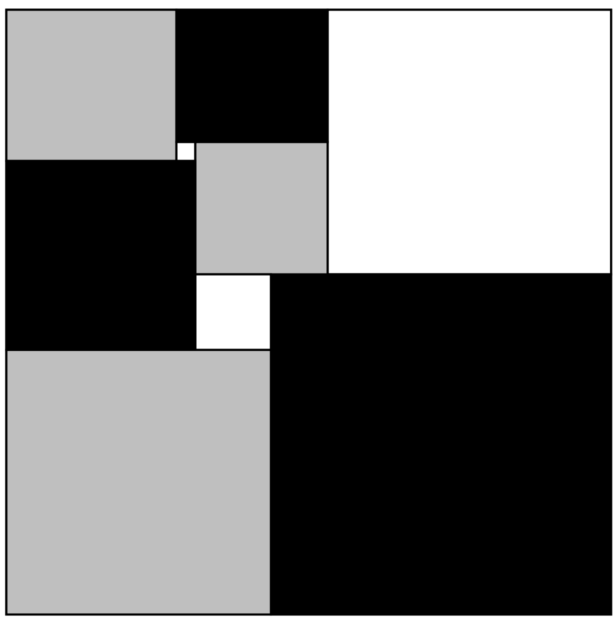
Problems and Solutions

Junior Division

Problem 1. You are given nine square tiles, with sides of lengths 1, 4, 7, 8, 9, 10, 14, 15 and 18 units, respectively. They can be used to tile a rectangle without gaps or overlaps.

Find the lengths of the sides of the rectangle, and show how to arrange the tiles.

Solution: The rectangle is 33×32, and one arrangement is shown below.



The area of the rectangle is equal to the sum of the areas of the squares, which is $1056 = 2^5 \times 3 \times 11$; one dimension is at least 33, and the other is at least 29, and the only factorisation of 1056 which allows this is 33×32.

Problem 2. Find all sets of three equally–spaced integers (positive or negative) whose product is 3240.

Solution: There are eight such sets, namely $\{12, 15, 18\}$, $\{6, 18, 30\}$, $\{3, 24, 45\}$, $\{-36, -3, 30\}$, $\{-30, -6, 18\}$, $\{-30, -9, 12\}$, $\{-36, -15, 6\}$ and $\{-81, -40, 1\}$.

Let the numbers be $x - y,\ x$ and $x + y$ with $y > 0$. It is not hard to show (see the solution of Senior Question 1) that if $x > 0$, then x is a divisor of 3240 with $15 \leq x \leq 40$,

$y^2 = x^2 - \dfrac{3240}{x}$, and we obtain the table

x	15	18	20	24	27	30	36	40
$x^2 - \frac{3240}{x}$	9	144	238	441	609	792	1206	1518
y	3	12	$-$	21	$-$	$-$	$-$	$-$
$x - y$	12	6		3				
x	15	18		24				
$x + y$	18	30		45				

while if $x < 0$, $x = -z$, then z is a divisor of 3240 with $z \leq 40$ and $y^2 = z^2 + \dfrac{3240}{z}$, and we obtain the table

z	1	2	3	4	5	6	8	9	10
$z^2 + \frac{3240}{z}$	3241	1624	1089	826	673	576	469	441	424
y	$-$	$-$	33	$-$	$-$	24	$-$	21	$-$
$x - y$			-36			-30		-30	
x			-3			-6		-9	
$x + y$			30			18		12	

z	12	15	18	20	24	27	30	36	40
$z^2 + \frac{3240}{z}$	414	441	504	562	711	849	1008	1386	1681
y	$-$	21	$-$	$-$	$-$	$-$	$-$	$-$	41
$x - y$		-36							-81
x		-15							-40
$x + y$		6							1

Problem 3. A rectangular room is paved with square tiles all the same size.

Show how you can draw a right-angled triangle on the floor with the following properties:

The vertices of the triangle are at corners of tiles, the hypotenuse lies along the edge of the room, and the ratio of the lengths of the shorter sides is $2 : 3$.

Can it be done if the ratio of the lengths of the shorter sides is $m : n$?

Solution: Suppose two sides of the room lie along the positive x and y axes, and that the square tiles have side length 1 unit. Join the points $O(0,0)$ and $P(0, m^2 + n^2)$ to the point $Q(mn, m^2)$. Then OQP is a right–angled triangle and $OQ : PQ = m : n$.

Slope of $OQ = \frac{m^2-0}{mn-0} = \frac{m}{n}$, slope of $PQ = \frac{m^2+n^2-m^2}{0-mn} = -\frac{n}{m}$,
length of $OQ = \sqrt{m^2n^2 + m^4} = m\sqrt{m^2+n^2}$,
length of $PQ = \sqrt{m^2n^2 + n^4} = n\sqrt{m^2+n^2}$.

Problem 4. In a regular hexagon, how many triangles are there with their vertices at vertices of the hexagon?

Of these, how many have their centroid (the point of concurrency of the medians) on a diagonal (a line joining vertices) of the hexagon?

Solution: There are $\binom{6}{3} = 20$ triangles with their vertices at vertices of the hexagon, and all of them have their centroids on a diagonal.

There are twelve triangles (two standing on each edge of the hexagon) with two vertices at adjacent vertices of the hexagon, and their third vertex at a vertex of the hexagon not adjacent to either of the other two vertices of the triangle; there are six triangles (one at each corner of the hexagon) with all three vertices at neighbouring vertices of the hexagon; and there are two triangles with their vertices at non–adjacent vertices of the hexagon.

If the vertices are labelled $A,\ B,\ C,\ D,\ E,\ F$, then the triangles are ACF, ADF, BDA, BEA, CEB, CFB, DFC, DAC, EAD, EBD, EBF, ECF, ABF, BCA, CDB, DEC, EFD, FAE, ACE and BDF. $\triangle ABF$ has its centroid on AD, $\triangle ACF$ has its centroid on BE, and $\triangle ACE$ has its centroid on BE, AD and CF.

Problem 5. The $*$–product of the integers a and b is defined by

$$a * b = ab + a + b = b * a.$$

(i) Show that the integer n has the $*$–factorisation $n = 0 * n$.

(ii) Show that the integer n also has the $*$–factorisation $n = (-2) * m$ for some integer m by finding suitable m.

(iii) Find all $*$–factorisations of 18 and of 19.

(iv) Find all integers n that have no $*$–factorisations other than those in which one of the $*$–factors is 0 or -2.

Solution:

(i) $0 * n = 0n + 0 + n = n$.

(ii) $(-2) * m = -2m - 2 + m = -m - 2 = n$ yields $m = -n - 2$. That is, $(-2) * (-n - 2) = n$.

(ii) $18 = 0 * 18 = (-2) * (-20)$.

$19 = 0 * 19 = 1 * 9 = 3 * 4 = (-2) * (-21) = (-3) * (-11) = (-5) * (-6)$.

(iv) $a * b = n$ is equivalent to $(a + 1)(b + 1) = n + 1$. This has only the two ‘trivial’ solutions if n is positive and $n + 1$ is prime or if n is negative and $n + 1$ is the negative of a prime.

Problem 6. At the inaugural meeting of a newly formed society, the following fact is observed. If A, B and C are any three members, and if A and B know each other and B and C know each other, then C knows no member other than B. Show that the members can be separated into two rooms so that no two people in the same room know each other.

Solution:

There are several categories of people: singletons, pairs and star–like clusters.

Singletons can go in either room.

Pairs, who know one another but no–one else, can be separated into the two rooms.

Those who know two or more other people and are the centres of star–like clusters can be put in either room, and the people they know can be put in the other.

Senior Division

Problem 1. Find all sets of three equally-spaced integers (positive or negative) whose product is 2160.

Solution: There are four such sets, namely $\{6, 15, 24\}$, $\{5, 16, 27\}$, $\{-30, -3, 24\}$ and $\{-30, -12, 6\}$.

Let the numbers be $x - y$, x and $x + y$ with $y > 0$. Then

$$x(x^2 - y^2) = 2160.$$

So x is a divisor of 2160, and

$$y^2 = x^2 - \frac{2160}{x}.$$

If $x > 0$, we require

$$x^2 - \frac{2160}{x} > 0.$$

Also $y^2 \leq (x-1)^2$, or,

$$x^2 - \frac{2160}{x} \leq (x-1)^2,$$

$$\frac{2160}{x} \geq 2x - 1.$$

So $$15 \leq x \leq 30.$$

So we find

x	15	16	18	20	24	27	30
$x^2 - \frac{2160}{x}$	81	121	204	392	486	649	828
y	9	11	–	–	–	–	–
$x - y$	6	5					
x	15	16					
$x + y$	24	27					

If $x < 0$, put $x = -z$. Then

$$y^2 = z^2 + \frac{2160}{z}.$$

Then z is a divisor of 2160 and $y^2 \geq (z+1)^2$, or,

$$\frac{2160}{z} \geq 2z + 1.$$

So $$z \leq 30.$$

So we find

z	1	2	3	4	5	6	8	9	10
$z^2 + \frac{2160}{z}$	2161	1084	729	556	457	396	334	321	316
y	$-$	$-$	27	$-$	$-$	$-$	$-$	$-$	$-$
$x - y$			-30						
x			-3						
$x + y$			24						

z	12	15	16	18	20	24	27	30
$z^2 + \frac{2160}{z}$	324	369	391	444	508	666	809	972
y	18	$-$	$-$	$-$	$-$	$-$	$-$	$-$
$x - y$	-30							
x	-12							
$x + y$	6							

Problem 2. Four football teams A, B, C and D play each other in a round robin tournament. Each pair of teams plays exactly one match, and the winners score 2 points, the losers 0. If the match is a draw, each team scores 1 point.

John switches on the radio just in time to hear the announcer say 'Team D came fourth. So no two teams scored the same number of points, and the only draw was in the game A versus B.'

John was disappointed because his favourite team was not even mentioned.

Find the placing of his favourite team, and the number of points they scored.

Solution: John's favourite team, C, came second with 4 points.

Altogether there are 12 points to be won and no team can score more than 6. A's and B's final scores are odd and at most 5, and are different, so are $\{5, 3\}$, $\{5, 1\}$ or $\{3, 1\}$. We will assume for the moment that A does better than B.

D's final score is less than all the others. So if B's score is 1, D's is 0. But then C's score is 6 or 8. Obviously 8 is impossible, but so is 6, since that implies C beat A, while A's score of 5 implies A beat C. So the only possibility is that A scores 5, B scores 3, C scores 4 and D scores 0. (Or switch A and B.)

Problem 3. Prove that if x and n are positive integers then $(x-1)^{n+2} + x^{2n+1}$ is divisible by $x^2 - x + 1$.

Solution: If $x = 1$, the result is obvious. If $x > 1$ then $x^2 - x + 1 > 1$, and we can calculate modulo $x^2 - x + 1$.

It is easy to show that as n increases, x^n cycles with period 6 through $x,\ x - 1,\ -1,\ -x,\ -x + 1,\ 1$, so x^{2n+1} (starting with $n = 1$) cycles with period 3 through $-1,\ -x + 1,\ x$, while $(x - 1)^{n+2}$ (also starting with $n = 1$) cycles with period 3 through $1,\ x - 1,\ -x$.

So $(x - 1)^{n+2} + x^{2n+1} \equiv 0 \pmod{x^2 - x + 1}$ for all $n \geq 1$.

(The result is also true for $n = 0$.)

Problem 4. The sum of three numbers in a geometric progression is 21, and the sum of their squares is 189.

Find the numbers.

Do the same if the sum is s and the sum of squares is t.

Solution: (i) The numbers are $\{3,\ 6,\ 12\}$.

Suppose the numbers are $a,\ ar,\ ar^2$. Then

$$a(1+r+r^2) = 21,$$
$$a^2(1+r^2+r^4) = 189.$$

It follows that

$$a^2(1+2r+3r^2+2r^3+r^4) = 441$$

and $$441(1+r^2+r^4) = 189(1+2r+3r^2+2r^3+r^4),$$

or, on simplification,

$$2r^4 - 3r^3 - r^2 - 3r + 2 = 0,$$

$$(r^2+r+1)(2r^2-5r+2) = 0,$$

$$r = 2 \text{ or } \frac{1}{2}.$$

The result now follows easily.

(ii) In the case

$$a(1+r+r^2) = s,$$

$$a^2(1+r^2+r^4) = t,$$

we find that the three numbers are

$$\frac{s}{1+r+r^2},\quad \frac{rs}{1+r+r^2},\quad \frac{r^2s}{1+r+r^2},$$

where r is either root of

$$(s^2-t)r^2 - (s^2+t)r + (s^2-t) = 0,$$

unless $s = t = 0$, in which case either all three numbers are 0 or are a, $a\omega$ and $a\omega^2$, where $\omega \neq 1$ is a cube root of unity, or $s^2 = t \neq 0$, in which case $r = 0$, and the numbers are $s,\ 0$ and 0.

Problem 5.

(a) Given an isosceles trapezium, with equal sides of length a, parallel sides of lengths b and c, and diagonal of length d, prove that

$$d^2 = a^2 + bc.$$

(b) Hence, or otherwise, find the (shortest) distance across the surface of the Earth from London (52ºN, 0ºE) to Sydney (35ºS, 152ºE), assuming the Earth is a sphere of circumference $40000\ Km$.

Solution:

(a) Draw in one diagonal. In one triangle we find

$$d^2 = a^2 + b^2 - 2ab\cos\alpha,$$

while in the other

$$d^2 = a^2 + c^2 + 2ac\cos\alpha.$$

It follows that

$$cd^2 + bd^2 = a^2b + a^2c + b^2c + bc^2,$$

or

$$(b + c)d^2 = (b + c)(a^2 + bc).$$

(b) According to the data, the distance from London to Sydney is $17100\ Km$ to three significant digits. (However, since the latitude of Sydney is actually just less than $34^{\text{o}}S$, a closer answer is $17000\ Km$.)

Let L be London, S be Sydney, let P be the point in the Pacific with coordinates (52ºN, 152ºE), A be the point in the Atlantic with coordinates (35ºS, 0ºE), and let R be the radius of the Earth in Km.

Then L and P are 152^{o} apart on a circle of radius $r = R\cos 52^{\text{o}}$,

and

$$b = 2r\sin 76^{\text{o}} = 2R\cos 52^{\text{o}}\sin 76^{\text{o}}.$$

Similarly,

$$c = 2R\cos 35^{\text{o}}\sin 76^{\text{o}}$$

and

$$a = 2R\sin 43.5^{\text{o}}.$$

Thus

$$\begin{aligned} d &= 2R\sqrt{\sin^2 43.5^{\text{o}} + \cos 35^{\text{o}}\cos 52^{\text{o}}\sin^2 76^{\text{o}}} \\ &= 2R\sin\theta/2 \end{aligned}$$

where θ is the angle subtended by LS at the centre of the Earth.

The distance of London from Sydney in Km is

$$\begin{aligned} \text{distance} &= R\theta \\ &= 2R\sin^{-1}\sqrt{\sin^2 43.5^{\text{o}} + \cos 35^{\text{o}}\cos 52^{\text{o}}\sin^2 76^{\text{o}}} \\ &= 17100 \end{aligned}$$

to three significant figures.

Problem 6. Let S be the set of all real numbers of the form

$$\frac{m+n}{\sqrt{m^2+n^2}}$$

where m and n are positive integers.

Prove that for every pair of numbers x and y in S there is a number z in S between x and y.

Solution: Suppose $x,\ y \in S$. Then we can write

$$x = \frac{m+n}{\sqrt{m^2+n^2}}, \quad y = \frac{p+q}{\sqrt{p^2+q^2}},$$

and we can assume $m \leq n, \quad p \leq q$.

Define θ and ϕ, $0 < \theta,\ \phi \leq \frac{\pi}{4}$, by

$$\sin\theta = \frac{m}{\sqrt{m^2+n^2}}, \ \sin\phi = \frac{p}{\sqrt{p^2+q^2}}.$$

Suppose $x < y$. Then

$$\sin\theta + \cos\theta < \sin\phi + \cos\phi,$$

so, since the function $\sin x + \cos x$ is increasing on $(0, \frac{\pi}{4})$, we have $\theta < \phi$. It follows that $\tan\theta < \tan\phi$. That is,

$$\frac{m}{n} < \frac{p}{q}.$$

Choose a rational between $\dfrac{m}{n}$ and $\dfrac{p}{q}$, say

$$\frac{m}{n} < \frac{m+p}{n+q} < \frac{p}{q},$$

define $\xi \in (0, \frac{\pi}{4})$ by

$$\tan\xi = \frac{m+p}{n+q}$$

and z by

$$z = \sin\xi + \cos\xi = \frac{m+p+n+q}{\sqrt{(m+p)^2+(n+q)^2}}.$$

Then $z \in S$ and

$$\tan\theta < \tan\xi < \tan\phi,$$

$$\theta < \xi < \phi,$$

$$\sin\theta + \cos\theta < \sin\xi + \cos\xi < \sin\phi + \cos\phi,$$

$$x < z < y.$$