Parabola Volume 43, Issue 2 (2007)

Solutions to Problems 1231–1240

Q1231 Given $a > 0$, prove that

$$\underbrace{\sqrt{a+\sqrt{a+\cdots+\sqrt{a}}}}_{n\text{ times}} < \frac{1+\sqrt{4a+1}}{2}.$$

ANS: Let

$$x_1 = \sqrt{a}, \quad x_2 = \sqrt{a+\sqrt{a}}, \quad \ldots, \quad x_n = \underbrace{\sqrt{a+\sqrt{a+\cdots+\sqrt{a}}}}_{n\text{ times}}.$$

Then $x_n > x_{n-1}$. Also $x_n^2 = a + x_{n-1}$. So $x_n^2 < a + x_n$ or $x_n^2 - x_n - a < 0$. This implies that $x_n \in (s_1, s_2)$ where s_1 and s_2 are two solutions of the quadratic equation $s^2 - s - a = 0$. These solutions are

$$s_1 = \frac{1-\sqrt{1+4a}}{2} \quad \text{and} \quad s_2 = \frac{1+\sqrt{1+4a}}{2}.$$

This proves the required inequality.

Q1232 Let a and c be two distinct real numbers and b be their arithmetic average (i.e. $b = (a+c)/2$). Find the condition on a and c so that $\sin^2 a$ and $\sin^2 c$ are distinct, and that $\sin^2 b$ is the arithmetic average of $\sin^2 a$ and $\sin^2 c$.

ANS: Note that

$$\begin{aligned}\sin^2 b = \frac{\sin^2 a + \sin^2 c}{2} &\iff \sin^2 b - \sin^2 a = \sin^2 c - \sin^2 b \\ &\iff (\sin b + \sin a)(\sin b - \sin a) \\ &\qquad = (\sin c + \sin b)(\sin c - \sin b). \qquad (1)\end{aligned}$$

Using the additional formulae in trigonometry we deduce that (1) is equivalent to

$$\begin{aligned}&4\sin\frac{a+b}{2}\cos\frac{b-a}{2}\cos\frac{a+b}{2}\sin\frac{b-a}{2} \\ &\qquad = 4\sin\frac{b+c}{2}\cos\frac{c-b}{2}\cos\frac{b+c}{2}\sin\frac{c-b}{2}\end{aligned}$$

or, with the help of the double angle formula for sine,

$$\sin(a+b)\sin(b-a) = \sin(b+c)\sin(c-b).$$

Since $b = (a+c)/2$ we have $b - a = c - b$; thus the above identity is equivalent to

$$\sin(b-a)[\sin(a+b) - \sin(b+c)] = 0.$$

If $\sin(b-a) = 0$ then $b - a = k\pi$ for $k = 0, \pm 1, \pm 2, \ldots$ But in this case $b = a + k\pi$ and $c = b + k\pi$, so that $\sin^2 a = \sin^2 b = \sin^2 c$. In order that these are distinct numbers a, b and c must satisfy $\sin(a+b) = \sin(b+c)$, implying

$$a + b = b + c + 2k\pi \quad \text{or} \quad a + b = \pi - b - c + 2k\pi, \quad k = 0, \pm 1, \pm 2, \ldots.$$

The first condition will result in $\sin^2 a = \sin^2 b = \sin^2 c$, so the required condition on a and c is $a + b = \pi - b - c + 2k\pi$, or $a + c = (2k+1)\pi/2$, $k = 0, \pm 1, \ldots$

Q1233 Prove that for any odd interger $n \geq 3$ and any $a \neq 0$ there holds

$$\left(1 + a + \frac{a^2}{2!} + \frac{a^3}{3!} + \cdots + \frac{a^n}{n!}\right)\left(1 - a + \frac{a^2}{2!} - \frac{a^3}{3!} + \cdots - \frac{a^n}{n!}\right) < 1.$$

ANS: Let

$$f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} \quad \text{and} \quad g(x) = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots - \frac{x^n}{n!}.$$

It is easy to see that

$$f'(x) = f(x) - \frac{x^n}{n!} \quad \text{and} \quad g'(x) = -g(x) - \frac{x^n}{n!}.$$

If $h(x) = f(x)g(x)$ then

$$\begin{aligned} h'(x) &= \left(f(x) - \frac{x^n}{n!}\right) g(x) + f(x)\left(-g(x) - \frac{x^n}{n!}\right) \\ &= -2\frac{x^n}{n!}\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{x^{n-1}}{(n-1)!}\right). \end{aligned}$$

Since n is odd we have

$$1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{x^{n-1}}{(n-1)!} > 0 \quad \text{for all } x,$$

so that $h'(x) > 0$ for all $x < 0$, and $h'(x) < 0$ for all $x > 0$. This implies $h(x) < h(0) = 1$ for all $x \neq 0$, proving the required inequality.

Q1234 (suggested by J. Guest, Victoria; edited)

Is the following statement true? "No prime number of type $10s + 1$ is a divisor of any number of the type $5^n + 1$, where s and n are positive integers." Give reason for your answer.

ANS: The statement is not true. In fact, take $p = 521$ which is easily seen to be prime and is of the type $10s + 1$ with s being 52. On the other hand, taking $n = 5$ we obtain $5^5 + 1 = 3126 = 6 \times 521$.

Q1235 (suggested by J. Guest, Victoria) As Jack and Tony went for a stroll they spotted a nice fruit shop. Jack decided to buy 7 bananas, 3 oranges and 5 plums, while Tony bought 5 bananas, 7 oranges and 3 plums. Jack spent \$3.49 for his purchase and Tony \$3.89. You are given that the bananas cost more than 20 cents each. Find the price of all three types of fruit.

ANS: Let the price of each banana be x cents, each orange be y cents, and each plum be z cents. Then we deduce

$$7x + 3y + 5z = 349 \tag{1}$$
$$5x + 7y + 3z = 389. \tag{2}$$

Let us now eliminate z between (1) and (2). This provides

$$2x + 13y = 449.$$

As the greatest common divisor of 2 and 13 is 1, we must expect a valid solution. Next divide by the smaller coefficient, i.e. 2, to arrive at

$$x + 6y = \frac{449 - y}{2}. \tag{3}$$

The left-hand side of (3) being a positive integer, $t = (449 - y)/2$ must be an integer. It follows that

$$y = 449 - 2t \quad \text{for some integer } t.$$

This, together with (3) and (1), yields

$$x = 13t - 2694 \quad \text{and} \quad z = 3572 - 17t.$$

Since x, y and z are positive integers, t must satisfy $207 < t < 210$, i.e. $t = 208$ or $t = 209$. If $t = 208$, the bananas cost 10 cents each, which we cannot accept by data. For $t = 209$ we arrive at the only permissible solution

$$x = 23 \text{ cents}, \quad y = 33 \text{ cents} \quad \text{and} \quad z = 19 \text{ cents}.$$

Q1236 Calculate the sum

$$6 + 66 + 666 + \cdots + \underbrace{66\cdots66}_{n\ 6\text{'s}}, \quad n \geq 1.$$

ANS: Let S be the sum. Then

$$\begin{aligned}
\frac{3}{2}S &= 9 + 99 + 999 + \cdots + \underbrace{99\cdots99}_{n\ 9\text{'s}} \\
&= (10-1) + (10^2-1) + (10^3-1) + \cdots + (10^n-1) \\
&= (10 + 10^2 + \cdots + 10^n) - \underbrace{(1 + 1 + \cdots + 1)}_{n \text{ times}} \\
&= \frac{10(10^n - 1)}{10 - 1} - n.
\end{aligned}$$

Therefore,

$$S = \frac{2}{3}\left(\frac{10}{9}(10^n - 1) - n\right).$$

Q1237 Prove that the numbers 49, 4489, 44489, ..., obtained by inserting 48 in the middle of the preceding number are all perfect squares.

ANS: The nth term of the sequence is

$$\begin{aligned} a_n &= \underbrace{44\cdots 44}_{n \text{ terms}}\underbrace{88\cdots 88}_{n-1 \text{ terms}}9 \\ &= \left(4\times 10^{2n-1} + 4\times 10^{2n-2} + \cdots + 4\times 10^n\right) \\ &\quad + \left(8\times 10^{n-1} + 8\times 10^{n-2} + \cdots + 8\times 10\right) + 8 + 1 \\ &= 4\times 10^n(10^{n-1} + \cdots + 1) + 8(10^{n-1} + \cdots + 1) + 1. \end{aligned}$$

The formula for the sum of a geometric progression gives

$$\begin{aligned} a_n &= \frac{4}{9}\times 10^n(10^n - 1) + \frac{8}{9}(10^n - 1) + 1 \\ &= \frac{4}{9}\times 10^{2n} + \frac{4}{9}\times 10^n + \frac{1}{9} \\ &= \left(\frac{2\times 10^n + 1}{3}\right)^2. \end{aligned}$$

It remains to show that $(2\times 10^n + 1)/3$ is an integer. This is easy to see because

$$2\times 10^n + 1 = 2\underbrace{00\cdots 00}_{n \text{ terms}} + 1 = 200\cdots 01,$$

which is divisible by 3.

Q1238 Let x, y and z be three positive numbers such that $x < y < z$. Prove that if $1/z$, $1/y$ and $1/x$ form an arithmetic progression then $z - x$, y, and $x - y + z$ are the lengths of the sides of a right-angled triangle.

ANS: First we note that $z - x > 0$ and $x - y + z > 0$. Since $1/z$, $1/y$ and $1/x$ form an arithmetic progression there holds

$$\frac{1}{y} = \frac{1}{2}\left(\frac{1}{x} + \frac{1}{z}\right) = \frac{x+z}{2xz},$$

implying $2xz = xy + yz$. Therefore,

$$\begin{aligned} (x - y + z)^2 &= x^2 + y^2 + z^2 - 2xy - 2yz + 2zx \\ &= x^2 + 2xz + z^2 + y^2 - 2(xy + yz) \\ &= x^2 + 2xz + z^2 + y^2 - 4xz \\ &= x^2 - 2xz + z^2 + y^2 \\ &= (x - z)^2 + y^2. \end{aligned}$$

By the Pythagorean theorem, $z - x$, y, and $x - y + z$ are the lengths of the sides of a right-angled triangle.

Q1239 Find all real numbers x and y satisfying

$$4^{\sin x} - 2^{1+\sin x}\cos(xy) + 2^{|y|} = 0.$$

ANS: The given equation can be rewritten as

$$\left(2^{\sin x} - \cos(xy)\right)^2 + \left(2^{|y|} - \cos^2(xy)\right) = 0. \tag{1}$$

Due to $\cos^2(xy) \le 1 \le 2^{|y|}$, there holds $2^{|y|} - \cos^2(xy) \ge 0$, and thus equation (1) is equivalent to the system

$$\begin{aligned} 2^{\sin x} - \cos(xy) &= 0 \\ 2^{|y|} - \cos^2(xy) &= 0 \end{aligned} \tag{2}$$

But

$$2^{|y|} - \cos^2(xy) = 0 \iff \left(2^{|y|} = 1 \quad \text{and} \quad \cos^2(xy) = 1\right) \iff y = 0.$$

With this value of y,

$$2^{\sin x} - \cos(xy) = 0 \iff \sin x = 0 \iff x = k\pi, k = 0, \pm1, \pm2, \dots.$$

Therefore, all values of x and y satisfying the given equation are

$$x = k\pi, k = 0, \pm1, \pm2, \dots, \quad \text{and} \quad y = 0.$$

Q1240 Prove that if $S(x) = ax^2 + bx + c$ is an integer when $x = 0$, $x = 1$ and $x = 2$, then $S(x)$ is an integer whenever x is an integer.

ANS: By substituting successively $x = 0$, $x = 1$ and $x = 2$ into $S(x)$ we deduce that c, $a + b + c$, and $4a + 2b + c$ are integers. As a consequence $a + b$ is an integer, which then implies that $2a$ is an integer (write $2a = (4a + 2b + c) - 2(a + b) - c$).

If $2a$ is an even integer, then a is an integer, and thus b is an integer, resulting in $S(x)$ being an integer for all integers x.

If $2a$ is an odd integer, then $a = k + 1/2$ for some integer k. Hence, due to $a + b = l$ being an integer, there holds

$$\begin{aligned} S(x) &= (k + \frac{1}{2})x^2 + (l - a)x + c \\ &= (k + \frac{1}{2})x^2 + (l - k - \frac{1}{2})x + c \\ &= kx^2 + (l - k)x + c + \frac{1}{2}x(x - 1). \end{aligned}$$

Since either x or $x - 1$ is even, $x(x - 1)/2$ is an integer, and so is $S(x)$ for any integer x.