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History of Mathematics: Physical Mathematics

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The cartoon reproduced below first appeared in *The New Yorker*. It so caught the fancy of the Mathematical Association of America that they acquired the rights to it; they in their turn extended them to *Function*, which is why we can reproduce it here. But what is wrong with the sign it depicts? The addition is correct!

What makes the cartoon funny is the incongruity of adding a date (measured in years), an altitude (measured in feet – remember this is America) and a population (determined by a headcount) all together. The three numbers are all measurements, but they measure quite different things, and it is quite meaningless to add them up.

Numbers derived from measurements are multiples of a standard called a 'unit'. Thus a length is given in metres, where a *metre* is the standard unit of length that we adopt. Similarly for other measurements. The units for the different measurements fall into two principal categories. Some units are regarded as *basic*, while others are *derived*.

Australia has embraced the *metric system*, or *Système International*, and the units adopted are the so-called SI units. The only countries not officially committed to this system are the USA, Liberia and Myanmar (Burma), although the UK, Canada and possibly other countries have failed to implement it very effectively. SI units are simple and easy to work with; computations with them are simpler than those involving the

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traditional system still in use in the USA. Moreover, they are widely adopted. All the same, there is nothing sacrosanct about them. Inconvenience aside, the US system works perfectly well.

At present, the *basic* SI units are:

- the kilogram, a unit of *mass* (M)
- the metre, a unit of *length* (L)
- the second, a unit of *time* (T)
- the degree Kelvin, a unit of *temperature* (Θ)
- the ampere, a unit of *electric current*
- the candela, a unit of *luminosity*
- the mole, a unit of *chemical concentration* (C)

In this article, I will not consider all of these, but will concentrate attention on the first three.

Two general points need to be made, however. There is nothing sacrosanct, as I have said, about the choice of the SI units. *Mathematically*, there is no reason to prefer them to the US system (clumsy though this may be). Indeed, the need for communication between researchers using different systems of units underlies the mathematical techniques I shall describe in the course of this article.

The second point is that the choice of which units are taken as basic is not sacrosanct either. I shall say rather less on this matter, but merely note that there is a current push to have the mole downgraded and removed from the list of basic units and regarded instead as derived. Some older systems regarded force as a fundamental unit, but such usage is now obsolete. We might also reflect that astronomers often find it convenient to measure distances in terms of times. The light-year is not an SI unit, but nonetheless it is widely used.

Now consider the situation of two separate observers using different units. Suppose that Observer A employs the SI, while Observer B makes use of the US units. We may set up a conversion table between the two systems. Here is an example:

1 mile = 1609.344 metres   1 pound = 0.4536 kilograms
1 hour = 3600 seconds.

This table allows interconversion between the two systems of measurement.

Moreover, we can use it to deduce conversion factors connecting derived units as well (speed for example). Suppose for example that a car covers a distance of 70 kilometres (70 000 metres) in 45 minutes. In SI units, it has taken a time of 2700 seconds, and so its speed is 25.9 metres per second. For an American, however, it has traveled 43.5 miles in 3/4 of an hour for a speed of 58 miles per hour.

This situation may readily be generalized. Suppose that Observer A measures a distance l_A and a time t_A in one set of units and Observer B measures the same distance and time as l_B and t_B in another. Suppose that the conversion table connecting the two systems reads $l_A = Ll_B$ and $t_A = Tt_B$. Then Observer A calculates a speed $\nu_A = l_A/t_A$ while Observer B calculates a speed $\nu_B = l_B/t_B$. We may readily deduce from these equations, however, that the ratio $V = \nu_A/\nu_B = L/T$. This simple deduction is summed up by saying that speed has the *dimensions* of distance over time.

The dimensions of a physical quantity are usually designated by the use of square brackets. So we write $[V] = L/T$, or LT^{-1}.

When it comes to the measurement of angles, a minor complication arises. The SI unit of angular measure is the radian, and the radian measure of an angle θ, say, is defined by placing its vertex at the centre of a circle of radius r and measuring s the length of the arc lying between its two arms. The ratio s/r is now the radian measure of the angle θ. But now notice that it makes no difference what unit of length we use. Just as long as both s and r are measured in the same terms, the same result will be obtained as the radian measure of the angle. (The use of degrees or grades for angular measures amounts to the use of different scales of measurement for radial and circumferential measures!) Thus the radian measure of an angle θ is given as $[\theta] = LL^{-1} = I$ (as it is usual to write). The radian may be regarded as a 'semi-basic unit'; there is exactly one other, the steradian, which will not be considered here.

What I am calling 'physical mathematics' is the systematic use of these ideas to explore physical laws. It is surprising how very much can be deduced about a physical system simply by analyzing the dimensions of the quantities involved. The usual term is 'dimensional analysis'. The basic idea is that all physical laws must result in the same fundamental formulae, whatever system of units is employed.

The fundamental assumptions are:

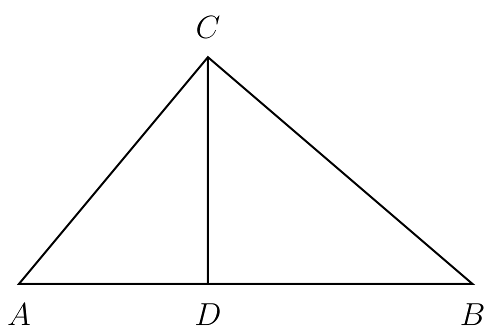
- The situation is described by a mathematical formula.
- This formula must be independent of the system of units employed.

The origins of dimensional analysis are somewhat obscure. There are anticipations of it in the geometry of similar figures, going all the way back to Euclid. Some of Galileo's insights are seen as early examples of dimensional arguments. Joseph Fourier (1768–1830) is another name often cited in this context. By the time of Osborne Reynolds (1842–1912) and James Clerk Maxwell (1831–1879), there was a well-developed theory, but not yet systematized. Two names are most commonly associated with the rise of the formal systematic theory. These are Aim Vaschy (1857–1899) and Edgar Buckingham (1867–1940). Vaschy probably gave the statement of the basic theorem sooner than did Buckingham, but Buckingham's statement was more influential and so now it bears his name.

The key result is that if a problem involves n measured quantities, and if these n involve r of the basic quantities, then (a few unusual cases aside) one can find $n - r$ combinations of the n, all different and all with dimension I. These combinations, Buckingham designated by the letter Π (the Greek equivalent of P and standing for 'product', as each was a product of powers of the original quantities). By 'different' he meant that no member of the set had a value that could be deduced from those of the others. The law connecting the original n quantities reduces to a formula connecting the $n - r$ 'dimensionless' products. This result is now known as *Buckingham's* Π-*Theorem*.

Where exactly one such dimensionless product is involved, then the theorem tells us this must be constant; where two are present, then one must be some function of the other; with three, one must be a function of the other two, and so on.

In order to make its statement clearer, I will give a few examples, starting with a simple one and working up to two more complicated ones. My first example is due to the Russian mathematician Grigory Barenblatt (1927-). It is a proof of Pythagoras' Theorem.



ABC is a triangle, right-angled at C. Write $|AB| = c$, $|BC| = a$, $|CA| = b$. As readers will know, Pythagoras' Theorem states that $c^2 = a^2 + b^2$. Barenblatt's proof involves drawing a line DC perpendicular to AB. He now considers the area of the triangle ABC. Call this area $\mathcal{A}$. Now $[\mathcal{A}] = L^2$, because area is measured in square length units.

The area will naturally depend on the length of the base, i.e. c. It will also depend on the position of the point C. But C lies on a semi-circle with AB as diameter, and so its position will be fixed by the value of the angle $BAC,\ \varphi$ let us call it. We thus have three quantities to consider: $\mathcal{A},\ c$ and φ. Thus $n = 3$. Of our list of basic units, only one is involved: length. So we have $r = 1$. Thus there are two products to consider. These can be found in various ways, but the simplest pair is $(\Delta c^{-2},\ \varphi)$. Thus the formula for the area involves only these two quantities. If we make $\mathcal{A}$ the subject of the formula, we must come up with $\Delta = c^2 f(\varphi)$ where $f(\varphi)$ where is an unknown function. (Actually $f(\varphi) = \frac{1}{4}\sin 2\varphi$, but we dont need to know this!)

Next Barenblatt considered the triangles ACD, BCD. We note that these are also right-angled triangles and that $\langle DAC = \langle BCD = \varphi$. Thus we can use the same formula to determine the areas of these two triangles. Let the areas be Δ_1 and Δ_2 respectively. We now have $\Delta_1 = b^2 f(\varphi),\ \Delta_2 = a^2 f(\varphi)$. But clearly, $\Delta = \Delta_1 + \Delta_2$, and so

$$c^2 f(\varphi) = b^2 f(\varphi) + a^2 f(\varphi),$$

and Pythagoras' Theorem follows.

Actually, this is a modern recasting of a very old proof, which you may well have seen. It is a version of a proof based on similar triangles or (equivalently) Trigonometry. This is a matter I leave to the reader to investigate further.

Geometric results from the Π-Theorem all use $r = 1$, as of all the basic units, only length is involved. Thus, all depend on the classical results and do not make use of the full power of the Π-Theorem. (This remark applies to the account I gave of Ptolemy's Theorem at the end of my previous article.)

However, if we move on to more complicated situations, then the wider scope of Dimensional Analysis becomes more evident. Take the case of the simple pendulum. Here we have a mass m swinging on the end of a rod, whose own mass is regarded as so small as not to matter. The rod has a length l and it swings through an angle θ under the influence of the force of gravity, represented by its weight mg, where g is a constant. The period of oscillation will be called τ. We write out all these quantities with their dimensions:

$$[m] = M,\ [l] = L,\ [\theta] = I,\ [\tau] = T,\ [g] = LT^{-2}.$$

Only the last of these requires any comment. But g is an acceleration, the acceleration that a heavy object experiences when it falls under the influence of gravity, acceleration is the time-rate of change of speed and hence the dimensions are as given.

Now, a glance at the list shows that m cannot be involved, because M occurs only in $[m]$ and so cannot be part of a dimensionless product. This leaves $n = 4$ and $r = 2$. So our formula must involve $4 - 2$, i.e. two dimensionless products. One is obvious: θ itself. The other is less so, but it can readily be checked that $[g\tau^2 l^{-1}] = I$. So our formula connects $g\tau^2 l^{-1}$ and θ. It is usual to make τ the subject of the formula, and so reach

$$\tau = f(\theta)\sqrt{\frac{l}{g}}$$

where $f(\theta)$ is some (unknown) function of θ. This is as far as this simple analysis can take us, but it already says a lot. A full treatment is very complicated, but it tells us (among other things) that when θ is small, $f(\theta)$ is approximately 2π.

My next example is perhaps the most spectacular example of the power of dimensional analysis. Although different accounts of it disagree over details, the principal lines of the story are clear. The US exploded the first atomic bomb in 1944 and made a film of the explosion. A copy of this film was later declassified and reached the British researcher G. I. Taylor. By analyzing the film frame by frame, he was able to deduce the energy released by the blast at a time when this figure was still top secret. What follows is an account giving the gist of part of his argument, although I am greatly oversimplifying his calculation.

Let R be the radius of the fireball generated by the explosion, t the time since detonation, E the energy released and ρ the density of the air outside the fireball (the density of the air inside the fireball he took to be very small, almost zero, because of the extreme heat).

We now have, using known results:

$$[R] = L,\ [t] = T,\ [E] = ML^2T^{-2},\ [\rho] = ML^{-3}.$$

There four quantities involved here, i.e. $n = 4$. We also have $r = 3$, because mass, length and time are all involved. We thus expect to find one dimensionless ratio, which can be taken to be $\Pi = E\rho R^{-5}t^2$. (Check, as an exercise, that this really is dimensionless.) The required law connecting R, t, E and ρ is thus of the form $\Pi = E\rho R^{-5}t^2 = C$, where C is a constant. This equation may be simplified to read
$R = At^{2/5}$, i.e. where A has the (constant) value $(E\rho C^{-1})^{1/5}$. Taking logarithms, Taylor found $\log R = \log A + \frac{2}{5}\log t$ and so he plotted $\log R$ against $\log t$ and indeed found a straight line with a slope $2/5$ as he had predicted. He now also knew the value of A and so, also knowing ρ, he was in the position to calculate E if only he knew the value of the constant C. Now Taylor could perhaps have deduced this from experiments with conventional explosives as some accounts suggest he did, but actually this approach involves some inaccuracies that he was not willing to incur. [An atomic bomb generates its explosion from a relatively small centre, which he approximated by a point. Conventional explosives, by contrast need a much more considerable amount of material to generate the fireball and the 'point-source' approximation is not a good one.] So rather he undertook a more detailed analysis that enabled him to calculate this constant. (It is very nearly 1.) He then knew the secret energy!

These three examples cover only a very small subset of the many uses to which the theory has been put. Readers will find many more in any one of the many books devoted to the subject. Barenblatt's *Dimensional Analysis* (English translation by P Makinen, published by Gordon & Breach, 1987) is one of his several books in the area. Others that can be recommended are Isaacson & Isaacson's *Dimensional Methods in Engineering and Physics* (Arnold, 1975), Kline's *Similitude and Approximation Theory* (McGraw-Hill, 1965), Langhaar's *Dimensional Analysis and Theory of Models* (Krieger, 1960), Sedov's *Similarity and Dimensional Methods* (Infosearch, 1959) and an older, but still excellent account, Focken's *Dimensional Methods and their Applications* (Arnold, 1953).

Taylor's calculation of the energy released by the atomic bomb was first published in the *Proceedings of the Royal Society* in 1950, and was republished in his *Collected Works* (Cambridge, 1971). It is also discussed by Barenblatt in the book detailed above, and also elsewhere. Another account is that by Bluman in the *International Journal of Mathematical Education in Science and Technology* (Vol. 14, 1983, pp. 259–272).

The promulgation of the systematic theory most influentially dates from a paper by Buckingham in *The Physical Review*, January 1900. Very many proofs of his Π-theorem have been published. One of the best is that by Brand in *Archive for Rational Mechanics and Analysis* (Vol. 1, 1957, pp. 35–45).

This simple list is only a sample of what is available in the literature. There are now many websites devoted to this material. A Google search conducted in mid-2007 revealed 103,000,000 of them! I looked at the first half dozen or so, starting with the Wikipedia entry. All of those I consulted were of excellent quality.