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Problems Section

You are invited to submit solutions to any or all of the following problems, accompanied by your name, (school and year if appropriate). Solutions to these problems will appear in the next issue of *Parabola*, and if received in time your solution(s) may be used.

Q1251 Show that the product of 4 consecutive integers is always one less than a perfect square.

Q1252 Given n distinct positive integers $a_1, a_2, \ldots, a_n$, none of which is divisible by a prime number greater than 3, prove that

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} < 3.$$

Q1253 Let p_0 be a prime number. We define recursively $p_k = p_{k-1} + 2k$, $k = 1, 2, \ldots$, and stop the recurrence when p_k is a composite number. The last index in the sequence, which depends on the initial value p_0, is denoted by $k(p_0)$. e.g. if $p_0 = 3$, then $p_1 = 5$ and $p_2 = 9$, so that $k(3) = 2$. For a given p_0, find an integer that $k(p_0)$ cannot exceed.

Q1254 Given a triangle ΔABC, construct a square $DEFG$ enclosed by the triangle such that D and E are on AB while F and G are on BC and AC, respectively.

Q1255 Let a and b be two real numbers satisfying $a + b \neq -1$ and $b \neq 0$. Show that if the equation

$$x^2 + ax + b = 0$$

has exactly one solution between 0 and 1, then the equation

$$\frac{1}{x+2} + \frac{a}{x+1} + \frac{b}{x} = 0$$

has exactly one positive solution.

Q1256 Assume that a and b are two integers such that $a^2 + b^2$ is divisible by 4. Show that a and b are both even.

Q1257 Let ABC be an acute-angled triangle with sides a, b and c, and altitudes h_a, h_b and h_c. Prove that

$$\frac{1}{2} < \frac{h_a + h_b + h_c}{a + b + c} < 1.$$

Q1258 A plane leaves a town of latitude $1°$S, flies x km due South, then x km due East, and x km due North. At this point, the plane is $3x$ km due East of the starting point. Find x.

Q1259 Prove that $\sqrt{2}$, $\sqrt{3}$ and $\sqrt{5}$ cannot be terms of an arithmetic progression.

Q1260 Show that $n^6 - n^2$ is divisible by 60 for any integer $n > 1$.