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History of Mathematics
A Proof (?) by Dedekind

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During my student days at the University of Melbourne I first encountered the passage I want to share with you. It was brought to my attention by a fellow-student, who found it interesting and unusual, as I did then and still do today. It occurs in a pamphlet entitled *The Nature and Meaning of Numbers*, now published in (a rather clumsy) English translation as Part 2 of a booklet *Essays on the Theory of Numbers* that first appeared in 1901. [The meaning of the term 'Theory of Numbers' usually applies to a branch of Algebra dealing with the study of the properties of whole numbers, primes and prime factorization; however, this is not the meaning here. Rather Dedekind, the author, is concerned to discuss the nature and meaning of numbers, just as he says.]

Richard Dedekind (1831–1916) is an interesting figure in the history of Mathematics. In previous columns in this series, I have discussed the introduction of the imaginary and complex numbers. Much earlier, however, was the introduction of irrationals; these together with the rational numbers, constitute the set of *real* numbers. But although it had been recognized from antiquity that numbers like $\sqrt{2}$ could not be expressed in rational terms, it took until the 19^{th} century before a truly satisfactory account of the reals was produced, and the version now most commonly given is that of Dedekind.

The passage in his pamphlet that so intrigued my fellow student is Section 66 (on p. 64 of the booklet). I reproduce it below, but I have taken the liberty of improving the quality of the English.

> Theorem. There exist infinite systems.
>
> Proof. My own realm of thoughts, i.e. the totality S of all things that can be objects of my thought, is infinite. For if s signifies an element of S, then the thought s, that s can be an object of my thought, is itself an element of S. If we regard s as a transform $\phi(s)$ of the element s, then the transform ϕ of S, thus determined, has the property that the transform S is part of S; and S is certainly a proper part of S, because there are elements in S (e.g., my own ego) which are different from the thought s and therefore not contained in S. Finally it is clear that if a, b are different elements of S, their transforms a, b are also different, and that therefore the transformation ϕ is a distinct (similar) transform. Hence S is infinite.

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Essentially, Dedekind is arguing that infinite systems can exist because he has given a real-life example of one. His 'proof' uses the property of infinite systems that allows an infinite set (here S) to be assigned a 1-1 correspondence with a proper part (S) of itself. [For example, the (infinite) set of positive integers can be placed in 1-1 correspondence with the (also infinite, but at first guess smaller) set of even numbers because in the sets $\{1, 2, 3, \cdots\}$ and $\{2, 4, 6, \cdots\}$ each member of the first set corresponds to precisely one member of the second and *vice versa*. Such paradoxical correspondences cannot occur with finite sets.]

My interest in this 'proof' was re-aroused recently. The Australian Council of Educational Research is in the process of producing a series of books on aspects of number-concepts. I wrote the second, concerned with linguistic origins of number-words. The first was written by my colleague John Crossley, who told how the concept of number has been developed and refined over the course of history. John's book, *Growing Ideas of Number*, discusses Dedekind's 'proof' on pp. 61, 62.

He presents it in a slightly more accessible form. I can think of a thing, T say, and then I can think of that thought, as a new thought T or $\phi(T)$, to use Dedekind's notations. Then I can think of this new thought and so generate a further new thought T or $\phi(\phi(T))$ and so on.

However, the matter is not really so clear. We are, of course, quite used to being able to apply Mathematics to the real world. For example we all use this aspect of Mathematics when we count objects, or balance our budget. Probably, we reach our concept of the number 2 (for example) by abstracting from our experience of pairs of objects. However, once we have grasped the notion, we deal with the number 2 directly, and attribute properties to it, more or less as if it possessed an independent existence. Mathematical notions like 2 can relate to real-life situations, but are seen as being distinct from those real-life situations. It is now usually accepted that although Mathematics can apply to the world of experience, the application cannot proceed in reverse. The Mathematics transcends the applications.

Dedekind's 'proof' is seen as an attempt to override this principle. But to accept this possibility is to say that the Mathematics *does not* transcend the instances on which it is based.

The key distinction is that between *necessary* and *contingent* truths. *Necessary* truths are those things that are true and could not possibly be otherwise; *contingent* truths are those things that *are* true, but need not be. The distinction has been well illustrated by the popular mathematics writer W. W. Sawyer in his book *Mathematician's Delight*:

> I can imagine zinc being dropped into sulphuric acid and nothing happening.
> But can you imagine twice two being five?

That mixing of zinc with sulphuric acid gives rise to a chemical reaction is an example of a *contingent* fact; $2 \times 2 \neq 5$, by contrast, is a *necessary* one.

This is the gist of an objection to Dedekind's 'proof' by Brouwer (1881–1966). We can apply Mathematics to everyday events, because necessary truths imply contingent ones, but the reverse process, which is what Dedekind is attempting, is invalid. In 1907, Brouwer wrote a thesis *On the Foundations of Mathematics*, which has since had a great influence on the development of Computer Science because of its insistence on the requirement that a mathematical proof should proceed by well-defined algorithms and should reach resolution in finite time. In this context, he wrote (in Chapter 3 of the thesis) that Dedekind's 'proof' (in its original form, as distinct from my subsequent paraphrase of it) constitutes an attempt to avoid the use of 'and so on'. However,

> In order to have mathematical significance, this system ought to have been completed by a mathematical existence proof. But in order to give that, we shall certainly be forced to use the intuition 'and so on', and then we see at once that we can obtain all the arithmetical theorems much more easily than by Dedekind's contrived system; accordingly Dedekind does not give the existence proof. He does give in §66 a proof for: [There exist infinite systems], but 1st a proof is needed for [this, and, what is more,] 2nd his proof, which introduces [my own realm of thoughts] is false because [my own realm of thoughts] cannot be viewed mathematically, so it is not certain that with respect to such a thing the ordinary axioms of whole and part will remain consistent. Consequently Dedekind's system has no mathematical significance; in order to give it logical significance, an independent consistency proof would be needed, but Dedekind does not give such a proof. If he had given it, he would necessarily have appealed to the intuition of 'and so on', and by recognizing this intuition, he would have seen that using it he could have constructed arithmetic in a very simple way, and from that moment on his logical system would have appeared to him gratuitous as well as cumbersome.

Crossley comments: "I agree, provided one understands 'Dedekind's system' as simply the one of his world of thoughts". My own reading, however, is that Brouwer intends his criticism to be more far-reaching, revealing a flaw in the entire enterprise that Dedekind has embarked upon. He seems to me to imply that the deficiency of this 'proof' is enough to invalidate the entire 'system' that Dedekind has built upon this basis. However, Crossley does not follow him in this and strikes me as having analyzed the matter better: essentially if we simply take 'There exist infinite systems' as an axiom, then all the rest of Dedekind's 'system' remains intact, although we might agree with Brouwer that it is needlessly complicated.

Dedekind himself notes that his 'proof' is not entirely new. In a footnote to it he acknowledges an indebtedness to an earlier mathematician, Bolzano (1781–1848). Bolzano regarded himself as a philosopher rather than as a mathematician, but nevertheless he is now celebrated for his mathematical achievements. His book *Paradoxes of the Infinite* was translated into English in 1950. In its Section 13, it includes the following discussion.

> . . . the next question to be asked . . . is whether there exist objects to which [the word *infinite*] can be applied . . . . This I venture to answer in a decided *affirmative*. Even in the *realm of things which do not claim actuality, and do not even claim possibility*, there exist beyond dispute sets which are infinite. *The set of all 'absolute propositions and truths'* is easily seen to be infinite.

> For if we fix our attention upon any truth taken at random, say the proposition that there exist such things as truths, or any other proposition, and label it 'A', we find that the proposition conveyed by the words 'A is true' is distinct from the proposition A itself, since it has the complete 'proposition A' for its own subject. Now by the same law which enabled us to derive from the proposition A another and different one, which we shall call B, we are further enabled to derive a third proposition C from B, and so forth without end. The aggregate of all these propositions, every one of which is related to its predecessor by having the latter for its own subject, and the latter's truth for its own assertion, comprises a set of members (each member a proposition) which is greater than any particular finite set. ... we can always continue the construction of such propositions — or rather, that such further propositions exist whether we construct them or not. Whence it follows that the aggregate of all the above propositions enjoys a multiplicity surpassing every individual integer, and is therefore infinite.

Where Dedekind used 'the thought of the thought', Bolzano used 'the truth of the truth'. Otherwise the arguments are the same. At first sight, the Bolzano version may strike us as the stronger one, because it does not seem to depend on the psychological task of thinking of thinking of thoughts, but rather relies on the truth of truth of truths. But the moment we realize that these truths have to be grasped by a human mind, we are placed firmly back in the psychological realm.

All the same, it is open to question quite how either author intended the argument to be understood. Dedekind is perhaps a little more explicit: to refer to 'the totality S of all things that *can* be objects of my thought' [my italics] is not quite the same as saying that I *do* think all these infinitely many thoughts. And how quite are we intended to understand 'can'? True, it is always possible to distinguish the thought T itself from the thought of that thought, T. That is to say, we can make an abstract differentiation between them. However, if T is sufficiently complicated a thought, it may not be possible to distinguish *in our actual mental processes* between T and T.

The thought of a thought, or the knowledge of a knowledge, has been the subject of various literary uses. Here are two of them.

The Victorian novelist Elizabeth Gaskell wrote in her novel *Cranford* (1863):

> [Mrs. Forrester, an impoverished gentlewoman] sat in state, pretending not to know what cakes were sent up; though she knew, and we knew, and she knew that we knew, and we knew that she knew that we knew, she had been busy all the morning making tea-bread and sponge-cakes.

The Danish writer Karen Blixen, writing under the pseudonym Pierre Andrézel, has in her 1946 novel *The Angelic Avengers*:

> They do not know for certain whether we have recognised them, whether we know that they are murderers. And even if, to be on the safe side, they reckon with it, they cannot be sure that we know that they know that we know that they are murderers!

In both instances, the word-play goes on just enough to achieve its effect, but any further elaboration would result in confusion.

Quite remarkably, this point had been made explicitly by another philosopher-mathematician, Gottfried Leibniz (1646–1716). [Leibniz is most famous as the co-discoverer, with Newton, of the calculus.] Leibniz was concerned to debate with the English philosopher John Locke (1632–1704) the matters raised in that author's book *Essay concerning Human Understanding*. Leibniz's response, *New Essays on Human Understanding* was written in 1704, the year of Locke's death. In deference to that event, it was not submitted for publication, and indeed did not see print until well after Leibniz's own death. It takes the form of a dialogue between two philosophers: Philalethes (who presents Locke's views) and Theophilus (who gives Leibniz's responses). In Volume II, Chapter 1, he has two passages relevant to our theme. I quote an English translation by Peter Remnant and Jonathan Bennett (Cambridge University Press, 1981). In §11, we read:

> Philalethes: It is 'hard to conceive that a thing should think, and not be conscious of it.'
>
> Theophilus: That is undoubtedly the crux of the matter — the difficulty by which many able people have been perplexed. But here is the way to escape from it. Bear in mind that we do think of many things all at once, but pay heed only to the thoughts that stand out most distinctly. That is inevitable; for if we were to take note of everything we should have to direct our attention to an infinity of things at the same time . . . .

Later in §19, we find a further elaboration of the theme. Theophilus says:

> '. . . it is impossible that we always reflect explicitly on all our thoughts; for if we did, the mind would reflect on each reflection, *ad infinitum*, without ever being able to move on to a new thought. For example, in being aware of some present feeling, I should have always to think about that feeling, and furthermore to think that I think of thinking about it, *ad infinitum*. It must be that I stop reflecting on all the reflections, and that eventually some thought is allowed to occur without being thought about; otherwise I would dwell forever on the same thing.'

What Leibniz asserts is that while we know things and know that we know that we know them, we do not, in any meaningful sense, continue this chain of knowledge beyond [a few] steps. He regards it as being self-evidently absurd to assert, as Dedekind and Bolzano do, that the process can be continued to infinity.

Leibniz, in contrast to Bolzano and Dedekind, is quite clearly addressing a psychological issue, and the truths of Psychology are to be seen as *contingent* truths, just as for Sawyer are those of Chemistry. But this feature is precisely what invalidates the 'proofs' that Bolzano and Dedekind advance. 'My own realm of thoughts' is an object in the real (contingent) world, and so cannot be used to validate a would-be necessary truth.

Dedekind's argument that infinite systems can exist because he has given a real-life example of one fails because of this confusion. Leibniz is surely right in rejecting the notion of our entertaining infinitely many thoughts all at once. However, Mathematics *needs* the infinite; it would be immeasurably the poorer without it. So we take it as an axiom, and proceed on this basis.

An interesting question to ask is whether either Bolzano or Dedekind had encountered the Leibniz discussion. It is the sort of thing that they well might have read. However, neither refers to it, so perhaps the odds are against this possibility. Moreover, I suspect that had they known, they would have felt compelled to answer the objection.