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Problems Section

You are invited to submit solutions to any or all of the following problems, accompanied by your name, (school and year if appropriate). Solutions to these problems will appear in the next issue of *Parabola*, and if received in time your solution(s) may be used.

Q1261 A hat contains $N = 2^n$ tickets, $n = 2, 3, 4, \ldots$, each marked with a number from 1 to N. (Each ticket has a different number.) In a game, players are asked to draw from the hat two tickets, read them, and replace them. Prize winners are those who draw two numbers whose ratio is 2. The money is refunded if the ratio is 2^k for $k = 2, 3, \ldots$. Find the probability of

1. a win;
2. a refund.

Q1262 Find all functions f satisfying

$$f\left(\frac{x-3}{x+1}\right) + f\left(\frac{x+3}{1-x}\right) = x \quad \text{for all } x \neq \pm 1. \tag{1}$$

Q1263 Two players A and B play a game with 100 marbles. They take turns to remove the marbles, at least 1 and at most 5 marbles each time. The player who removes the last marbles remaining wins the game.

1. Find a strategy for A to win if he starts the game.
2. Who will lose if initially there are 102 marbles on the table, and both players know the correct strategy?

Q1264 David wants to draw 50 points on a square of side length 14cm so that any two points are separated by at least 3cm. Can he do so?

Q1265 Let a, b, c, and d be positive constants, and x and y be real numbers such that

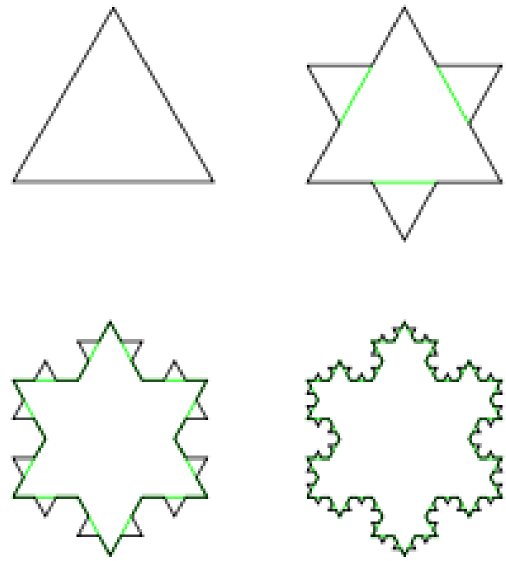
$$\sin^2 x + \cos^2 y \neq 0 \quad \text{and} \quad \sin^2 y + \cos^2 x \neq 0.$$

Prove that

$$\frac{a+b}{c+d} \leq \frac{a\sin^4 x + b\cos^4 y}{c\sin^2 x + d\cos^2 y} + \frac{a\cos^4 x + b\sin^4 y}{c\cos^2 x + d\sin^2 y} \leq \frac{a}{c} + \frac{b}{d}.$$

Find the conditions on x and y such that equalities occur.

Q1266 Let C_1 be an equilateral triangle of side length a cm. We define the curves C_2, C_3, C_4, ... successively from the previous one by trisecting each side and adding an equilateral triangle to the middle section of it, as shown in the figure.



1. Find the perimeter P_n of C_n.
2. Find the area A_n of the region bounded by C_n.
3. What happens when n gets infinitely large?

Q1267 Find all positive integers m and n satisfying

$$m^3 + n^3 = m^4.$$

Q1268 How many integers are of the form $a_1a_2a_3\cdots a_{n-1}a_na_{n-1}\cdots a_3a_2a_1$, where $0 < a_1 < a_2 < a_3 < \cdots < a_{n-1} < a_n$ and $n \geq 2$?

e.g. 272, 34843, and 135676531.

Q1269 A polyhedron is a solid with planar polygons as its faces. Are there convex polyhedra which have 7 edges?

Q1270 Prove that if A, B and C are three angles of a triangle ΔABC then

$$\frac{\sin^2 A + \sin^2 B + \sin^2 C}{\cos^2 A + \cos^2 B + \cos^2 C} \leq 3.$$

When does the equality occur?