Parabola Volume 44, Issue 1 (2008)

Solutions to Problems 1251–1260

Q1251 Show that the product of 4 consecutive integers is always one less than a perfect square.

ANS: We can denote the 4 consecutive integers by $n-1$, n, $n+1$ and $n+2$. Then

$$\begin{aligned}(n-1)n(n+1)(n+2) &= [n(n+1)][(n-1)(n+2)]\\ &= (n^2+n)(n^2+n-2)\\ &= [(n^2+n-1)+1][(n^2+n-1)-1]\\ &= (n^2+n-1)^2-1.\end{aligned}$$

Q1252 Given n distinct positive integers $a_1, a_2, \ldots, a_n$, none of which is divisible by a prime number greater than 3, prove that

$$\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}<3.$$

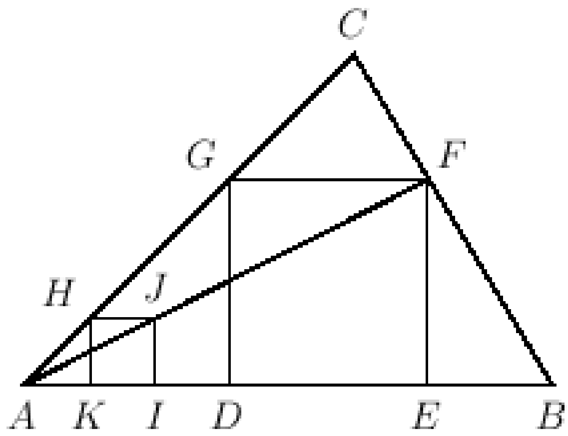
ANS: Since no integer a_i, $i=1,\ldots,n$, is divisible by a prime number greater than 3, each integer a_i can be expressed in the form $a_i = 2^{p_i}3^{q_i}$ where p_i and q_i are non-negative integers. The given integers a_i being distinct, we have $(p_i,q_i)\neq(p_j,q_j)$ if $i\neq j$. Assume that there are only r distinct values of p_i, namely $p_1, \ldots, p_r$, $1\le r\le n$. (For those pairs (p_i,q_i) having the same value p_i, the values of q_i are all different.) We can partition all n pairs (p_i,q_i) into r groups, in each group the p_i's are the same but the q_i's are different. Then

$$\begin{aligned}\sum_{i=1}^{n}\frac{1}{a_i} &= \sum_{i=1}^{n}\frac{1}{2^{p_i}3^{q_i}}\\ &= \frac{1}{2^{p_1}}\sum_{\text{group }1}\frac{1}{3^{q_i}}+\cdots+\frac{1}{2^{p_r}}\sum_{\text{group }r}\frac{1}{3^{q_i}}\\ &< \frac{1}{2^{p_1}}\sum\frac{1}{3^{q_i}}+\cdots+\frac{1}{2^{p_r}}\sum\frac{1}{3^{q_i}},\end{aligned}$$

where the sums on the last row of the right-hand side are taken over all different values of q_i. Therefore,

$$\sum_{i=1}^{n}\frac{1}{a_i}<\left(\sum_{i=1}^{r}\frac{1}{2^{p_i}}\right)\left(\sum\frac{1}{3^{q_i}}\right)=\left(\sum\frac{1}{2^{p_i}}\right)\left(\sum\frac{1}{3^{q_i}}\right),$$

where the first summation on the right hand side is taken over all distinct values of p_i, and the second is taken over all distinct values of q_i. These distinct values of p_i and q_i



being nonnegative integers, we deduce that

$$\sum_{i=1}^{n} \frac{1}{a_i} < \left(\sum_{k=0}^{\infty} \frac{1}{2^k}\right)\left(\sum_{l=0}^{\infty} \frac{1}{3^l}\right) = \frac{1}{1-1/2} \times \frac{1}{1-1/3} = 3.$$

Q1253 Let p_0 be a prime number. We define recursively $p_k = p_{k-1} + 2k$, $k = 1, 2, \ldots$, and stop the recurrence when p_k is a composite number. The last index in the sequence, which depends on the initial value p_0, is denoted by $k(p_0)$. E.g., if $p_0 = 3$, then $p_1 = 5$ and $p_2 = 9$, so that $k(3) = 2$. For a given p_0, find an integer that $k(p_0)$ cannot exceed.

ANS: (correct answers submitted by John Barton, Victoria)

Comment: It is easy to see that

$$p_k = p_0 + 2(1 + 2 + \cdots + k) = p_0 + k(k+1).$$

With $k = p_0 - 1$ we have $p_k = p_0^2$, which is a composite number. So $k(p_0) \leq p_0 - 1$. Note that $k(7) = 2 < 7 - 1$.

A little arithmetical exploration, beginning with, say, 11 or 41, throws up the suggestion that it could be profitable to explore the question: "What happens to our selected prime p_0, if we add $p_0 - 1$ terms of the arithmetical progression 2, 4, 6, 8 ...? " The sum $2 + 4 + 6 + 8 + \cdots |(2p_0 - 2)$ is easily seen, by writing the terms in reverse order, to be $p_0(p_0 - 1)$. Adding this to p_0 we get p_0^2, a composite number. Thus $k(p_0)$ cannot exceed $p_0 - 1$.

(11 & 41 are examples of primes in which $k(p_0) = p_0 - 1$)

Q1254 Given a triangle ΔABC, construct a square $DEFG$ enclosed by the triangle such that D and E are on AB while F and G are on BC and AC, respectively.

Construct the square $DEFG$ as follows:

1. Choose a point H on AC;
2. Draw the square $HKIJ$;

3. Produce AJ to meet BC at F;
4. Draw $EF \perp AB$, $FG \parallel AB$, and $DG \perp AB$.

$DEFG$ is the required square.

Q1255 Let a and b be two real numbers satisfying $a + b \neq -1$ and $b \neq 0$. Show that if the equation

$$x^2 + ax + b = 0$$

has exactly one solution between 0 and 1, then the equation

$$\frac{1}{x+2} + \frac{a}{x+1} + \frac{b}{x} = 0$$

has exactly one positive solution.

ANS: Let $f(x) = x^2 + ax + b$. Since the equation

$$x^2 + ax + b = 0$$

has exactly one solution between 0 and 1, $f(0)$ and $f(1)$ are of opposite sign, implying

$$f(0)f(1) = b(1 + a + b) < 0. \tag{1}$$

Now rewrite the equation

$$\frac{1}{x+2} + \frac{a}{x+1} + \frac{b}{x} = 0$$

as

$$(1 + a + b)x^2 + (1 + 2a + 3b)x + 2b = 0. \tag{2}$$

The discriminant is

$$\Delta = (1 + 2a + 3b)^2 - 8b(1 + a + b).$$

Inequality (1) yields $\Delta > 0$, so that (2) has exactly 2 solutions α and β satisfying

$$\alpha\beta = -\frac{2b}{1 + a + b} < 0$$

due to (1). So there is exactly one positive solution.

Q1256 Assume that a and b are two integers such that $a^2 + b^2$ is divisible by 4. Show that a and b are both even.

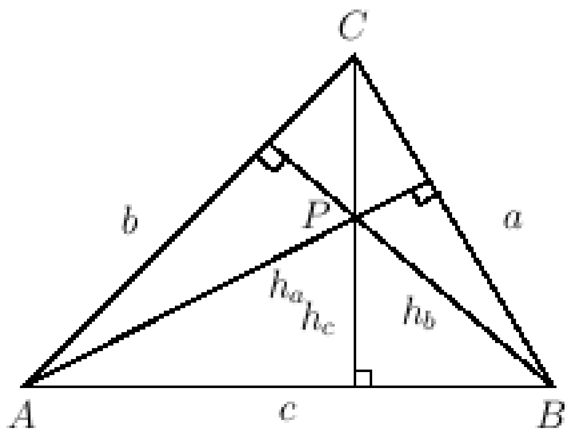
ANS: Assume that a is odd, i.e., $a = 2k + 1$ for some integer k. Then since $a^2 + b^2$ is divisible by 4 we have

$$a^2 + b^2 = 4l \quad \text{for some integer } l,$$

implying

$$b^2 = 4l - (2k + 1)^2 = 4(l - k^2 - k - 1) + 3. \tag{3}$$

Now if b is even then b^2 is divisible by 4, contradicting (3). If b is odd, i.e., $b = 2m + 1$ then $b^2 = 4m^2 + 4m + 1$ which also contradicts (3). Therefore a cannot be odd. Similar argument implies b is even.



Q1257 Let ABC be an acute angled triangle with sides a, b and c, and altitudes h_a, h_b and h_c. Prove that

$$\frac{1}{2} < \frac{h_a + h_b + h_c}{a + b + c} < 1.$$

ANS:

The altitudes of ΔABC are concurrent at the orthocentre P, which lies within ΔABC because the triangle is acute-angled. Therefore,

$$PA + PB > c, \quad PB + PC > a, \quad PC + PA > b,$$

which implies

$$2(PA + PB + PC) > a + b + c.$$

On the other hand, $PA < h_a$, $PB < h_b$ and $PC < h_c$, so that

$$\frac{a + b + c}{2} < h_a + h_b + h_c,$$

i.e.,

$$\frac{h_a + h_b + h_c}{a + b + c} > \frac{1}{2}.$$

Moreover,

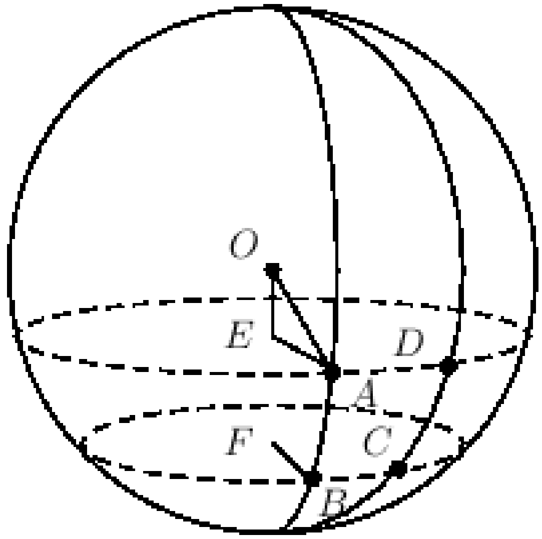
$$h_a < b, \quad h_b < c, \quad h_c < a,$$

implying $h_a + h_b + h_c < a + b + c$,

$$\text{i.e.,} \qquad \frac{h_a + h_b + h_c}{a + b + c} < 1.$$

Q1258 A plane leaves a town of latitude 1°S, flies x km due South, then x km due East, and x km due North. At this point, the plane is $3x$ km due East of the starting point. Find x.

ANS:



Let A be the starting point, B be the point x km due S of A, C be x km due E of B, and D be x km due N of C. Since the difference in longitude between A and D is equal to that between B and C, and since the arc AD of the small circle centred at E is three times the arc BC of the small circle centred at F, the radius EA is three times the radius FB. Since A is of latitude 1°S, the angle $\angle OAE = 1°$. So $EA = OA\cos 1°$, which implies

$$FB = \frac{1}{3}OA\cos 1° = OA\cos\theta°,$$

where θ is the latitude of B. It follows that

$$\cos\theta° = \frac{1}{3}\cos 1°,$$

which implies

$$\theta = \cos^{-1}(\frac{1}{3}\cos 1°) = 70.53°,$$

so that $\angle AOB = \theta - 1 = 69.53°$. Hence,

$$x = \text{length of the arc } AB = OA \times \angle AOB \text{ (in radian)} = \frac{\pi}{180} \times 69.53 \times OA.$$

Taking the radius of the earth to be $6,367$ km we obtain $x \approx 7,727$ km.

Q1259 Prove that $\sqrt{2}$, $\sqrt{3}$ and $\sqrt{5}$ cannot be terms of an arithmetic progression.

ANS: Assume that $\sqrt{2}$, $\sqrt{3}$ and $\sqrt{5}$ are terms of an arithmetic progression. Let d be the common difference. Then $\sqrt{3} = \sqrt{2} + kd$ and $\sqrt{5} = \sqrt{2} + ld$ for some nonzero integers k and l. It follows that

$$\frac{\sqrt{3}-\sqrt{2}}{k} = d = \frac{\sqrt{5}-\sqrt{2}}{l}.$$

Rearranging we obtain

$$\sqrt{2}(k-l) = k\sqrt{5} - l\sqrt{3},$$

and by squaring both sides

$$2(k-l)^2 = 5k^2 + 3l^2 - 2\sqrt{15}kl.$$

Therefore,

$$\sqrt{15} = \frac{5k^2 + 3l^2 - 2(k-l)^2}{2kl},$$

which is a contradiction as $\sqrt{15}$ is irrational while the right hand side is rational.

Q1260 Show that $n^6 - n^2$ is divisible by 60 for any integer $n > 1$.

ANS: Note that $n^6 - n^2 = n^2(n^4 - 1) = n^2(n^2+1)(n-1)(n+1)$. It suffices to show that $n^6 - n^2$ is divisible by 3, 4 and 5.

- Since $n-1$, n and $n+1$ are three consecutive integers, $n^6 - n^2$ is divisible by 3.
- If n is even then n^2 is divisible by 4, and so is $n^6 - n^2$. If n is odd, namely $n = 2k+1$, then $n^2 - 1 = 4k^2 + 4k$, so $n^6 - n^2$ is divisible by 4.
- To prove that $n^6 - n^2$ is divisible by 5 we consider the following cases:
  1. If $n = 5l$ then clearly $n^6 - n^2$ is divisible by 5.
  2. If $n = 5l + 1$ then $n - 1 = 5l$ and thus $n^6 - n^2$ is divisible by 5.
  3. If $n = 5l + 2$ or $n = 5l + 3$ then $n^2 + 1$ is divisible by 5 and so is $n^6 - n^2$.
  4. If $n = 5l + 4$ then $n + 1$ is divisible by 5 and so is $n^6 - n^2$.

  All cases are exhausted, so $n^6 - n^2$ is divisible by 5.