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Surface Area of a Rectangular Prism

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In my first year of teaching, I was given a formula summary sheet to be handed out to my year 11 general mathematics class. One formula was particularly appealing.

$$2(xy + yz + xz)$$

This is the formula for the **surface area** of the rectangular prism with dimensions x, y, z. However my mind went back to other formulae that I learnt in high school. The expansion of $(x + y + z)^2$ is used in evaluating the sum of the squares of the roots of cubic equations.

$$(x + y + z)^2 = x^2 + y^2 + z^2 + 2(xy + yz + xz)$$

Rearranging the above identity yields

$$2(xy + yz + xz) = (x + y + z)^2 - (x^2 + y^2 + z^2). \tag{1}$$

Equation (1) shows that if we know the **sum of the squares** as well as the **square of the sum** of the length, width and height of a rectangular prism, its surface area is determined. This is an interesting property of the surface area of a rectangular prism. Unfortunately equation (1) would not have any practical use because in real life we always know the actual dimensions of the rectangular prism before the sum of the squares of the rectangular prism. However equation (1) has an important theoretical application which involves using formulae from another branch of mathematics: statistics.

Recall that the formula for the **mean** or the **expected value** $E(X)$ of a set of scores is

$$E(X) = \frac{1}{n}\sum_{i=1}^{n} x_i.$$

Therefore the mean of the dimensions of the rectangular prism is

$$E(X) = \frac{x + y + z}{3}.$$

If we square both sides of the equation above and rearrange we get

$$(x + y + z)^2 = 9E(X) \tag{2}$$

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Also the mean of the squares of the dimensions of the rectangular prism is

$$E(X^2) = \frac{x^2 + y^2 + z^2}{3}.$$

Therefore we can express the sum of the squares of the dimensions of the rectangular prism in terms of $E(X)$

$$x^2 + y^2 + z^2 = 3E(X) \tag{3}$$

Moreover the **variance** $Var(X)$ of the set of scores is given by

$$Var(X) = E(X^2) - E(X)^2.$$

Furthermore the variance is equal to the square of the standard deviation.

$$Var(X) = \sigma^2$$

Now we can express the surface area of the rectangular prism in terms of the mean of the dimensions of the rectangular prism $E(X)$ and the standard deviation of the dimensions of the rectangular prism σ. Therefore using (2) and (3),

$$\begin{aligned} 2(xy + yz + xz) &= (x + y + z)^2 - (x^2 + y^2 + z^2) \\ &= 9E(X)^2 - 3E(X^2) \\ &= 6E(X)^2 + 3E(X)^2 - 3E(X^2) \\ &= 6E(X)^2 - 3(E(X^2) - E(X)^2) \\ &= 6E(X)^2 - 3Var(X) \\ &= 6\overline{X}^2 - 3\sigma^2 \end{aligned}$$

The above analysis shows that the surface area of the rectangular prism is determined by the mean and the standard deviation of the dimensions of the rectangular prism.

If we are given a straw to build a skeleton for a rectangular prism, then the mean of the dimensions of the rectangular prism must be fixed because it is always equal to twelve times the total length of the straw. The above formula shows that to maximise the surface area of the rectangular prism, the standard deviation must be minimised. The smallest value standard deviation can take is zero. This occurs precisely when the dimensions of the rectangular prism are the same. Hence the rectangular prism with the highest surface area must be a cube.

Of course the problem of maximising $2(xy + yz + xz)$ subject to the constraint $x + y + z = P$ (where P is a constant) can be solved routinely by several variables calculus using the technique of Lagrange Multipliers. However the above algebraic proof explains why spreading the dimensions of the rectangular prism far apart will result in a smaller surface area. It also shows that a seemingly unrelated branch of mathematics can shed light on another branch of mathematics.